Homework #12

MATH 126 — Calculus II
Monday, Nov 23

Please print your name:

Problem 1. (9.6.2, 9.6.6) Determine if the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3/2}}$

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+5}{n^2+4}$

Solution.

(a) This series converges absolutely, because $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a converging p-series. In particular, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3/2}}$ converges (because absolute convergence implies convergence).

Alternatively, we can argue that this series converges by the alternating series test. (This is just one of the alternating p-series we discussed in class.) Note that it answers the question, but reveals less information than our previous argument: we still don't know if the series converges absolutely or only converges conditionally.

(b) The series diverges because the terms $(-1)^{n+1} \frac{n^2+5}{n^2+4}$ do not converge to zero. □

Problem 2. (9.7.4, 9.7.14) Find the series' radius and interval of convergence. For what values of x does the series converge absolutely, for what values does it converge conditionally?

(a) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

(b) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$

Solution.

(a) We apply the ratio test with $a_n = \frac{(3x-2)^n}{n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n}\right| = |3x-2| \frac{n}{n+1} \to |3x-2| \text{ as } n \to \infty$$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$ converges for all x such that $|3x-2| < 1$ or, equivalently, $\left|x - \frac{2}{3}\right| < \frac{1}{3}$.

The radius of convergence therefore is $\frac{1}{3}$. We already know that the series converges if $\left|x - \frac{2}{3}\right| < \frac{1}{3}$, that is, if $x \in \left(\frac{2}{3} - \frac{1}{3}, \frac{2}{3} + \frac{1}{3}\right) = \left(\frac{1}{3}, 1\right)$. We also know that the series diverges if $\left|x - \frac{2}{3}\right| > \frac{1}{3}$. What we don't know yet is whether the series converges at the endpoints of the interval. We still need to think about the cases $x = \frac{1}{3}$ and $x = 1$:

- $x = 1$: in that case, the series is $\sum_{n=1}^{\infty} \frac{(3 \cdot 1 - 2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series, which we know diverges.

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- $x=\frac{1}{3}$: in that case, the series is $\sum_{n=1}^{\infty} \frac{\left(3 \cdot \frac{1}{3}-2\right)^n}{n}=\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. This is the alternating harmonic series. It converges by the alternating series test, but it does not converge absolutely. Hence, for $x=\frac{1}{3}$, our series converges conditionally.

The exact interval of convergence therefore is $\left[\frac{1}{3}, 1\right)$.

Moreover, we know that the series converges conditionally for $x=\frac{1}{3}$, and it converges absolutely for $x \in \left(\frac{1}{3}, 1\right)$. (Note that the ratio test works with absolute values, so we always get absolute convergence from it.)

(b) We apply the ratio test with $a_n=\frac{(x-1)^n}{n^3 3^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(x-1)^{n+1}}{(n+1)^3 3^{n+1}} \cdot \frac{n^3 3^n}{(x-1)^n}\right|=\left|\frac{x-1}{3}\right| \frac{n^3}{(n+1)^3} \rightarrow \left|\frac{x-1}{3}\right| \text{ as } n \rightarrow \infty$$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$ converges for all x such that $\left|\frac{x-1}{3}\right|<1$ or, equivalently, $|x-1|<3$.

The radius of convergence therefore is 3. We already know that the series converges if $|x-1|<3$, that is, if $x \in (1-3, 1+3)=(-2,4)$. We also know that the series diverges if $|x-1|>3$. What we don't know yet is whether the series converges at the endpoints of the interval. We still need to think about the cases $x=-2$ and $x=4$:

- $x=4$: in that case, the series is $\sum_{n=1}^{\infty} \frac{(4-1)^n}{n^3 3^n}=\sum_{n=1}^{\infty} \frac{1}{n^3}$. This is a p-series with $p=3$, which we know converges (absolutely, because all the terms are positive).

- $x=-2$: in that case, the series is $\sum_{n=1}^{\infty} \frac{(-2-1)^n}{n^3 3^n}=\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$. This is an alternating p-series. It converges absolutely because $p=3>1$.

The exact interval of convergence therefore is $[-2,4]$.

Moreover, we know that the series converges absolutelyfor all $x \in [-2,4]$. (Note that the ratio test works with absolute values, so we always get absolute convergence from it.) □

Problem 3. (9.8.2) Find the Taylor polynomials of orders 0, 1, 2 and 3 generated by $f(x)=\sin x$ at $x=0$.

Solution. Since $\sin(x)=\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots,$ the Taylor polynomials of orders 0, 1, 2 and 3 are

$$0, \quad x, \quad x, \quad x-\frac{x^3}{6}.$$

□

Problem 4. (9.9.2) Using substitution, find the Taylor series of $e^{-x/2}$ at $x=0$.

Solution. Since $e^x=\sum_{n=0}^{\infty} \frac{x^n}{n!}$, it follows that $e^{-x/2}=\sum_{n=0}^{\infty} \frac{(-x/2)^n}{n!}=\sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n} x^n$. □

Problem 5. Find the first four terms of the Taylor series of $e^x \cos(x)$ at $x=0$.

Solution. Since $e^x=\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\cos x=\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, we have

$$e^x \cos(x)=\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots\right)\left(1+0x-\frac{x^2}{2}+0x^3+\ldots\right)=1+x-\frac{x^3}{3}+\ldots$$

[The first four terms of a Taylor series at $x=0$ are the terms $a_0+a_1x+a_2x^2+a_3x^3$. When computing the product, we therefore only included terms up to x^3. If we want to also include terms up to x^4, that is, compute $a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$, then we get

$$e^x \cos(x)=\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\ldots\right)\left(1-\frac{x^2}{2}+\frac{x^4}{24}+\ldots\right)=1+x-\frac{x^3}{3}-\frac{x^4}{6}+\ldots]$$ □

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