Practice for Midterm #1

MATH 125 — Calculus 1
Tuesday, Feb 5

Please print your name:

Besides the allowed calculator, no notes or tools of any kind will be permitted.

Problem 1. Have another look at the homework, especially those problems that you struggled with.

Problem 2. Retake the three quizzes! (Versions with and without solutions are posted to our course website.)

Problem 3. Go through the lecture sketches (posted to our course website) and do the problems we did in class (ignore the solutions until you have solved the problem yourself).

Problem 4. Determine the following limits (or state that they don't exist).

(a) $\lim\limits_{x\to -1}\dfrac{x+3}{x+1}$

(b) $\lim\limits_{x\to\infty}\tan^{-1}(x)$

(c) $\lim\limits_{x\to -\infty}\sin(x)$

(d) $\lim\limits_{x\to -\infty}\dfrac{\sin(x)}{1+x^2}$

(e) $\lim\limits_{x\to\infty}\dfrac{2+\sqrt{x}-2x^3}{5-x^3}$

(f) $\lim\limits_{x\to 0}\dfrac{\sin(2x)}{\sin(x)}$

(g) $\lim\limits_{x\to 0}\dfrac{\sin(2x)}{\sqrt{x}}$

(h) $\lim\limits_{x\to\infty}\dfrac{\sin(2x)}{\sqrt{x}}$

Solution.

(a) $\lim\limits_{x\to -1}\dfrac{x+3}{x+1}$ does not exist because $\lim\limits_{x\to -1^-}\dfrac{x+3}{x+1}=\dfrac{2}{0^-}=-\infty$ and $\lim\limits_{x\to -1^+}\dfrac{x+3}{x+1}=\dfrac{2}{0^+}=\infty$

(b) $\lim\limits_{x\to\infty}\tan^{-1}(x)=\dfrac{\pi}{2}$

(c) $\lim\limits_{x\to -\infty}\sin(x)$ does not exist

(d) $\lim\limits_{x\to -\infty}\dfrac{\sin(x)}{1+x^2}=0$ (by the sandwich theorem because $-\frac{1}{1+x^2}\leqslant\frac{\sin(x)}{1+x^2}\leqslant\frac{1}{1+x^2}$ and $\pm\frac{1}{1+x^2}\overset{x\to\infty}{\longrightarrow}0$)

(e) $\lim\limits_{x\to\infty}\dfrac{2+\sqrt{x}-2x^3}{5-x^3}=\dfrac{-2}{-1}=2$

(f) $\lim\limits_{x\to 0}\dfrac{\sin(2x)}{\sin(x)}=2$ (recall that $\sin(x)\approx x$ for small x, so that $\frac{\sin(2x)}{\sin(x)}\approx\frac{2x}{x}=2$)

[Using limit laws: $\lim\limits_{x\to 0}\dfrac{\sin(2x)}{\sin(x)}=\lim\limits_{x\to 0}2\dfrac{\frac{\sin(2x)}{2x}}{\frac{\sin(x)}{x}}=2\cdot\dfrac{1}{1}=2$]

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(g) $\lim_{x\to 0} \frac{\sin(2x)}{\sqrt{x}} = 0$ (again, easiest to see by recalling that $\sin(x) \approx x$ for small x, so that $\frac{\sin(2x)}{\sqrt{x}} \approx \frac{2x}{\sqrt{x}} = 2\sqrt{x}$)

[Using limit laws: $\lim_{x\to 0} \frac{\sin(2x)}{\sqrt{x}} = \lim_{x\to 0} 2\sqrt{x}\, \frac{\sin(2x)}{2x} = 2 \cdot 0 \cdot 1 = 0$]

(h) $\lim_{x\to\infty} \frac{\sin(2x)}{\sqrt{x}} = 0$ (by the sandwich theorem because $-\frac{1}{\sqrt{x}} \leqslant \frac{\sin(2x)}{\sqrt{x}} \leqslant \frac{1}{\sqrt{x}}$ and $\pm\frac{1}{\sqrt{x}} \xrightarrow{x\to\infty} 0$) $\square$

Problem 5. Determine the following limits. [Show work!]

(a) $\lim_{x\to\infty} \left(x^2 - \sqrt{x^4 + 5x + 1}\right)$.

(b) $\lim_{x\to\infty} \left(x^2 - \sqrt{x^4 + 5x^2 + 1}\right)$.

(c) $\lim_{x\to\infty} \left(x^2 - \sqrt{x^4 + 5x^3 + 1}\right)$.

Solution. We proceed in the same way for each case:

(a) $$x^2 - \sqrt{x^4+5x+1} = \frac{\left(x^2 - \sqrt{x^4+5x+1}\right)\left(x^2 + \sqrt{x^4+5x+1}\right)}{x^2 + \sqrt{x^4+5x+1}} = \frac{-5x-1}{x^2+\sqrt{x^4+5x+1}}$$
$$= \frac{-\frac{5}{x} - \frac{1}{x^2}}{1 + \sqrt{1 + \frac{5}{x^3} + \frac{1}{x^4}}} \xrightarrow{x\to\infty} -\frac{-0-0}{1+\sqrt{1+0+0}} = 0$$

(b) $$x^2 - \sqrt{x^4+5x^2+1} = \frac{\left(x^2 - \sqrt{x^4+5x^2+1}\right)\left(x^2 + \sqrt{x^4+5x^2+1}\right)}{x^2 + \sqrt{x^4+5x^2+1}} = \frac{-5x^2-1}{x^2+\sqrt{x^4+5x^2+1}}$$
$$= \frac{-5 - \frac{1}{x^2}}{1 + \sqrt{1 + \frac{5}{x^2} + \frac{1}{x^4}}} \xrightarrow{x\to\infty} -\frac{-5-0}{1+\sqrt{1+0+0}} = -\frac{5}{2}$$

(c) $$x^2 - \sqrt{x^4+5x^3+1} = \frac{\left(x^2 - \sqrt{x^4+5x^3+1}\right)\left(x^2 + \sqrt{x^4+5x^3+1}\right)}{x^2 + \sqrt{x^4+5x^3+1}} = \frac{-5x^3-1}{x^2+\sqrt{x^4+5x^3+1}}$$
$$= \frac{-5x - \frac{1}{x^2}}{1 + \sqrt{1 + \frac{5}{x} + \frac{1}{x^4}}} \xrightarrow{x\to\infty} -\frac{-\infty-0}{1+\sqrt{1+0+0}} = -\infty$$ $\square$

Problem 6. For what values of a and b is $f(x) = \begin{cases} \sin(x) - a, & x < 0, \\ a\sqrt{x+1} + b, & 0 \leqslant x \leqslant 3 \\ x + 2b, & x > 3, \end{cases}$ continuous at every x? [Show work!]

Solution. Observe that $f(x)$ is always continuous at every point except, possibly, $x = 0$ and $x = 3$.

- At $x = 0$ we have:
  - $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (\sin(x) - a) = -a$
  - $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (a\sqrt{x+1} + b) = a + b = f(0)$

Hence, $\lim_{x\to 0} f(x) = f(0)$ (meaning that $f(x)$ is continuous at $x=0$) if and only if $-a = a+b$.

- At $x=3$ we have:
  - $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (a\sqrt{x+1}+b) = 2a+b = f(3)$
  - $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x+2b) = 3+2b$

  Hence, $\lim_{x\to 3} f(x) = f(0)$ (meaning that $f(x)$ is continuous at $x=3$) if and only if $2a+b = 3+2b$.

Taken together, this means that $f(x)$ is continuous everywhere if and only if $-a = a+b$ as well as $2a+b = 3+2b$.

Solving these two equations (the first gives $b=-2a$, which plugged into the second results in $0=3-4a$; hence, $a=\frac{3}{4}$ and $b=-2a=-\frac{3}{2}$), we find that $a=\frac{3}{4}$ and $b=-\frac{3}{2}$ are the only values that make $f(x)$ continuous. $\square$

Problem 7. [Show work!]

(a) Determine $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ for $f(x) = x^2-3x$.

(b) Determine $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ for $f(x) = \sqrt{2x+1}$.

Solution.

(a) Since $f(x+h) = (x+h)^2 - 3(x+h) = x^2+2hx+h^2-3x-3h$, we have

$$\frac{f(x+h)-f(x)}{h} = \frac{2hx+h^2-3h}{h} = 2x+h-3 \quad \xrightarrow{h\to 0} \quad 2x-3.$$

(b) Here, we have

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{2x+2h+1}-\sqrt{2x+1}}{h} = \frac{(2x+2h+1)-(2x+1)}{h(\sqrt{2x+2h+1}+\sqrt{2x+1})} = \frac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}},$$

so that $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}} = \frac{2}{\sqrt{2x+0+1}+\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$. $\square$

Problem 8. Let $f(x)$ be a complicated continuous function taking the following values:

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-1	2	4	1	-1	-3	4	4	1

How many solutions to the equation $f(x)=3$ can we guarantee? How about $f(x)=5$? Or $f(x)=4$?

Solution.

(a) By the intermediate value theorem, there must be a solution to $f(x)=3$ in $[-3,-2]$, in $[-2,-1]$, in $[1,2]$ and in $[3,4]$. We can therefore guarantee 4 solutions.

(b) We cannot guarantee any solutions to $f(x)=5$.

(c) We know that $x=-2$, $x=2$ and $x=3$ are solutions to $f(x)=4$, but the intermediate value theorem doesn't guarantee any additional solutions. We can therefore guarantee 3 solutions. □

Problem 9. Let $f(x)=x+\sqrt{\ln(x)}$.

(a) At which points is $f(x)$ continuous?

(b) Show that there is a number c in the interval $[1,5]$ such that $f(c)=3$.

Solution.

(a) Since $f(x)$ is constructed from functions which are continuous everywhere, we know that $f(x)$ is continuous at every point of its domain. Hence, $f(x)$ is continuous for all $x\geqslant 1$.

(b) $f(1)=1$ and $f(5)=6.2686$. Hence $f(1)<3<f(5)$ and the intermediate value theorem guarantees a c in $(1,5)$ such that $f(c)=3$. □

Problem 10. Consider the function $f(x)=\begin{cases} x, & \text{if } x\leqslant -1,\\ x^2, & \text{if } -1<x\leqslant 0,\\ x\sin\left(\frac{1}{x^2}\right), & \text{if } x>0.\end{cases}$

(a) Find the limit $\lim\limits_{x\to 0^+} f(x)$.

(b) At which points is $f(x)$ discontinuous?

Solution.

(a) Note that $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} x\sin\left(\dfrac{1}{x^2}\right)$. Because $-1\leqslant\sin\left(\frac{1}{x^2}\right)\leqslant 1$ we have $-x\leqslant x\sin\left(\frac{1}{x^2}\right)\leqslant x$.

Since $\lim\limits_{x\to 0^+} \pm x=0$, it therefore follows from the sandwich theorem that $\lim\limits_{x\to 0^+} x\sin\left(\dfrac{1}{x^2}\right)=0$.

(b) The only points where $f(x)$ is potentially discontinuous are $x=-1$ and $x=0$.

- $(x=0)$ Clearly, $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-} x^2=0$ and $f(0)=0$. By the first part, $\lim\limits_{x\to 0^+} f(x)=0$.

  It follows that $f(x)$ is continuous at $x=0$.

- $(x=-1)$ Here, we have

$$\begin{aligned} \lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} x=-1,\\ \lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} x^2=1,\\ f(-1) &= -1. \end{aligned}$$

  These values don't agree so $f(x)$ is discontinuous at $x=-1$. □