Example 74. Determine $\frac{\mathrm{d}}{\mathrm{d}x}x^x$.

Solution. (using logarithmic differentiation) Let $y = x^x$. Then $\ln(y) = x\ln(x)$. Differentiating both sides, we obtain

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \cdot \ln(x) + x \cdot \frac{1}{x} = 1 + \ln(x) \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = y(1 + \ln(x)) = x^x(1 + \ln(x)).$$

Solution. (equivalent alternative) Write $x^x = e^{\ln(x^x)} = e^{x\ln(x)}$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x}x^x = \frac{\mathrm{d}}{\mathrm{d}x}e^{x\ln(x)} = e^{x\ln(x)}\frac{\mathrm{d}}{\mathrm{d}x}[x\ln(x)] = e^{x\ln(x)}(1 + \ln(x)) = x^x(1 + \ln(x)).$$

Comment. Make sure that you can see that the two approaches are actually doing the exact same.

Derivatives of inverse trig functions

- $\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}$ **(derivative of $\sin^{-1}(x)$)**
- $\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}(x) = -\frac{1}{\sqrt{1 - x^2}}$ **(derivative of $\cos^{-1}(x)$)**
- $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = \frac{1}{1 + x^2}$ **(derivative of $\tan^{-1}(x)$)**

Example 75. (derivative of $\tan^{-1}(x)$)

Solution. Recall (or, rederive it!) that $\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x)$. It follows from $\tan(\tan^{-1}(x)) = x$ that

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan(\tan^{-1}(x)) = \sec^2(\tan^{-1}(x))\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = 1 \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = \frac{1}{\sec^2(\tan^{-1}(x))} = \frac{1}{1 + x^2}.$$

For the last step, recall that $\sec^2(x) = 1 + \tan^2(x)$. Therefore, $\sec^2(\tan^{-1}(x)) = 1 + \tan^2(\tan^{-1}(x)) = 1 + x^2$.

Example 76. (derivative of $\sin^{-1}(x)$)

Solution. It follows from $\sin(\sin^{-1}(x)) = x$ that

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(\sin^{-1}(x)) = \cos(\sin^{-1}(x))\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) = 1 \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1 - x^2}}.$$

For the last step, recall that $\cos^2(x) + \sin^2(x) = 1$ implies $\cos(x) = \pm\sqrt{1 - \sin^2(x)}$. Here, x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (because that's the domain for $\sin^{-1}(x)$) and we know that $\cos(x) \geqslant 0$ for those x. Hence, here, $\cos(x) = \sqrt{1 - \sin^2(x)}$ and $\cos(\sin^{-1}(x)) = \sqrt{1 - \sin^2(\sin^{-1}(x))} = \sqrt{1 - x^2}$.

Derivative of $\cos^{-1}(x)$. Repeat the same argument to derive that $\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}(x) = -\frac{1}{\sqrt{1 - x^2}}$.

Comment. Why did we have to pay more attention to getting the sign right here than for $\tan^{-1}(x)$? To see that, make a sketch of the curves (not graphs of functions!) $x = \sin(y)$ and $x = \tan(y)$. Then highlight the parts of those curves that are the graph of $\sin^{-1}(x)$ and $\tan^{-1}(x)$ (see sketch on next page). Recall that we had to make a choice when defining these inverse trig functions! We could have made other choices; try to see how these other choices would affect the derivative of $\sin^{-1}(x)$ and $\tan^{-1}(x)$. (The upshot is that other reasonable choices would change the sign of the derivative of $\sin^{-1}(x)$, but that the derivative of $\tan^{-1}(x)$ would be unaffected.)

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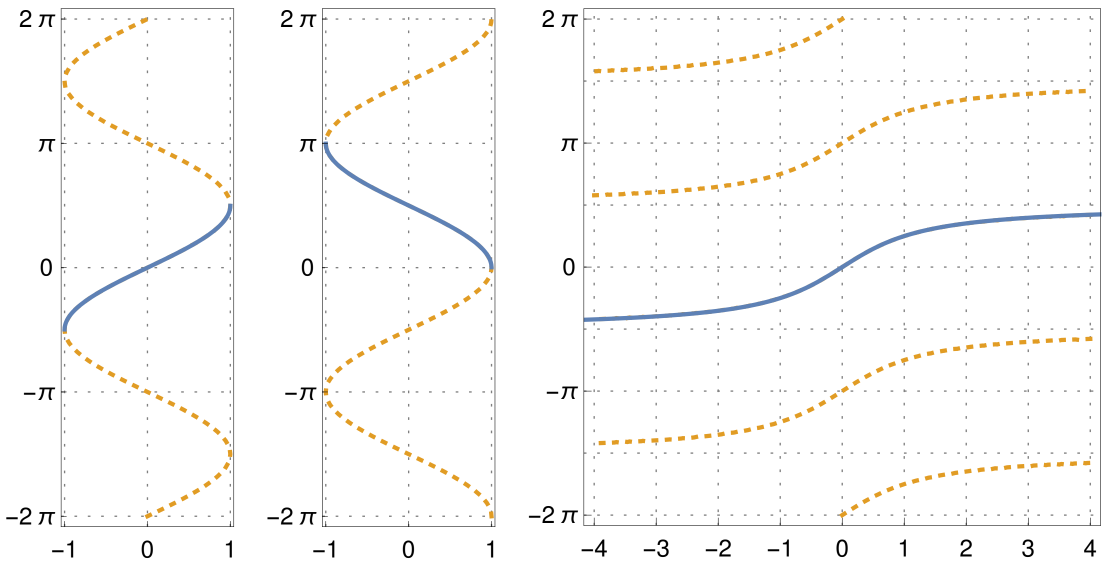
The curves $x = \sin(y)$, $x = \cos(y)$, $x = \tan(y)$ in dotted orange, and the choice of inverse function in solid blue:



Example 77. Determine $\frac{\mathrm{d}}{\mathrm{d}x}[x^4 \tan^{-1}(2x^7)]$.

Solution. $\frac{\mathrm{d}}{\mathrm{d}x}[x^4 \tan^{-1}(2x^7)] = 4x^3 \tan^{-1}(2x^7) + x^4 \frac{14x^6}{1 + (2x^7)^2} = 4x^3 \tan^{-1}(2x^7) + \frac{14x^{10}}{1 + 4x^{14}}$

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