Quiz #2 (Tuesday)

- read off limits from a graph
- compute a limit algebraically

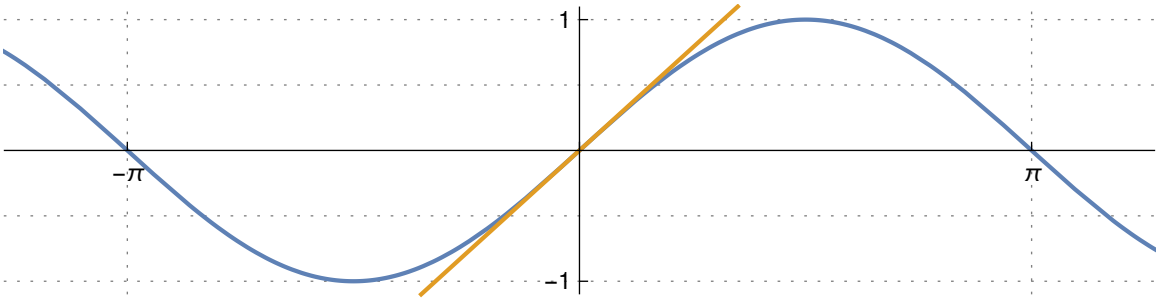
Example 21. Determine $\lim_{x\to 0} \frac{\sin(x)+1}{3\cos(x)+x}$.

Solution. There is no problem at $x=0$, so that we can just plug in $x=0$: $\lim_{x\to 0} \frac{\sin(x)+1}{3\cos(x)+x} = \frac{\sin(0)+1}{3\cos(0)+0} = \frac{1}{3}$.

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

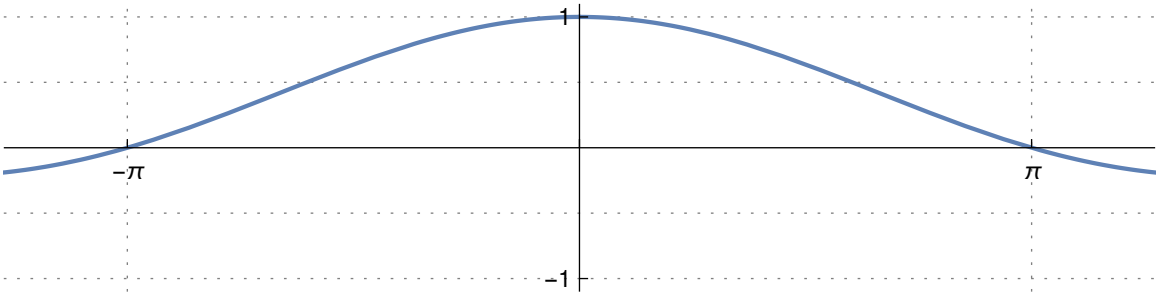
Note. This limit is a precise way of saying that $\sin(x) \approx x$ for small x.

Graph both $f(x)=\sin(x)$ and $g(x)=x$ together (for instance, using the website `http://desmos.com`) and observe how the graphs are becoming indistinguishable as you zoom in on $x=0$.



We will later learn tools that allow us to algebraically derive that limit easily. For an argument using trigonometry, check out the proof after Theorem 7 in Chapter 2.4 of our book.

Another way to look at it. Sketch the function $f(x)=\frac{\sin(x)}{x}$.



Note how the sketch suggests that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

Moreover, it suggests that $f(x)$ is nice (continuous!) everywhere, including at $x=0$, if we define $f(0)=1$.

Example 22. Determine $\lim_{x\to 0} \dfrac{3x}{\sin(4x)}$.

Solution. Note that $\dfrac{3x}{\sin(4x)} = \dfrac{3}{4} \cdot \dfrac{1}{\frac{\sin(4x)}{4x}}$.

Hence, $\lim_{x\to 0} \dfrac{3x}{\sin(4x)} = \dfrac{3}{4} \cdot \dfrac{1}{\lim_{x\to 0} \frac{\sin(4x)}{4x}} = \dfrac{3}{4} \cdot \dfrac{1}{1} = \dfrac{3}{4}$.

[Exactly which of the limit laws have we used?!]

Note. Setting $x = 4z$ in $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$ and noticing that $x \to 0$ is the same as $z \to 0$ produces $\lim_{z\to 0} \dfrac{\sin(4z)}{4z} = 1$.

Comment. Once we are comfortable with limits we can argue as follows:
For small x, we have $\sin(x) \approx x$ and hence $\sin(4x) \approx 4x$ (because $4x$ is small), so that $\dfrac{3x}{\sin(4x)} \approx \dfrac{3x}{4x} = \dfrac{3}{4}$.
Making this argument precise results in our earlier manipulations of limits.

Example 23. Determine $\lim_{x\to 0} \dfrac{\sin(3x)}{\sin(4x)}$.

Solution. Note that $\dfrac{\sin(3x)}{\sin(4x)} = \dfrac{3}{4} \cdot \dfrac{\frac{\sin(3x)}{3x}}{\frac{\sin(4x)}{4x}}$.

Hence, $\lim_{x\to 0} \dfrac{\sin(3x)}{\sin(4x)} = \dfrac{3}{4} \cdot \dfrac{\lim_{x\to 0} \frac{\sin(3x)}{3x}}{\lim_{x\to 0} \frac{\sin(4x)}{4x}} = \dfrac{3}{4} \cdot \dfrac{1}{1} = \dfrac{3}{4}$.

Example 24. Determine $\lim_{x\to 0} \dfrac{\tan(3x)}{\sin(4x)}$.

Solution. Note that $\dfrac{\tan(3x)}{\sin(4x)} = \dfrac{1}{\cos(3x)} \cdot \dfrac{\sin(3x)}{\sin(4x)}$.

It therefore follows from the previous example that $\lim_{x\to 0} \dfrac{\tan(3x)}{\sin(4x)} = \dfrac{1}{\lim_{x\to 0} \cos(3x)} \cdot \lim_{x\to 0} \dfrac{\sin(3x)}{\sin(4x)} = \dfrac{1}{1} \cdot \dfrac{3}{4} = \dfrac{3}{4}$.