Supercongruences for polynomial analogs of the Apéry numbers

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$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$

- Apéry numbers and their siblings
- supercongruences
- q-analogs

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

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THM (Apéry '78): $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q **Beukers, Zagier**

Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

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Q **Beukers, Zagier** Are there other tuples (a,b,c) for which the solution defined by $u_{-1}=0$, $u_0=1$ is integral?

- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions (with generating functions

$$ {}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\1,1\end{matrix}\right|4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\left.\begin{matrix}\alpha,1-\alpha\\1\end{matrix}\right|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2,$$

    with $\alpha = \frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4\cdot 3^3$)
  - 6 sporadic solutions

- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2+an+b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$ (Cooper)

The six sporadic Apéry-like numbers

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Modularity of Apéry-like numbers

- The Apéry numbers $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

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$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

Supercongruences for Apéry numbers

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THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$
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EG Simple combinatorics proves the congruence
$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$
For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,
$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$


Robert Osburn
(University of Dublin)


Brundaban Sahu
(NISER, India)

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers.

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

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$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Beukers, Coster '87-'88
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '16
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open (modulo p^3 Amdeberhan–Tauraso '16)
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	Osburn–Sahu–S '16
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	Gorodetsky '18

Non-super congruences are abundant

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(G)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{"points of period } n\text{"}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

In fact, up to a positivity condition, (G) characterizes realizability.

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- $a(n) = \operatorname{trace}(M^n)$ Jänichen '21, Schur '37; also: Arnold, Zarelua

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- (G) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

Basic q-analogs

- The natural number n has the q-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}$$

In the limit $q \to 1$ a q-analog reduces to the classical object.

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- The q-factorial:

$$[n]_q! = [n]_q \, [n-1]_q \cdots [1]_q$$

- The q-binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!} = \binom{n}{n-k}_q$$

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1 + q + q^2 + q^3 + q^5)(1 + q + q^2 + q^3 + q^4)}{1 + q}$$

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$$\begin{aligned}\binom{6}{2}_q &= \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q} \\ &= (1-q+q^2)\underbrace{(1+q+q^2)}_{=[3]_q}\underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}\end{aligned}$$

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$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$
$$= \underbrace{(1-q+q^2)}_{=\Phi_6(q)} \underbrace{(1+q+q^2)}_{=[3]_q} \underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

- The cyclotomic polynomial $\Phi_6(q)$ becomes 1 for $q=1$ and hence invisible in the classical world

The coefficients of q-binomial coefficients

- Here's some q-binomials in expanded form:

EG

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\begin{aligned}\binom{9}{3}_q = {} & q^{18} + q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} \\ & + 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 \\ & + 4q^4 + 3q^3 + 2q^2 + q + 1\end{aligned}$$

- The degree of the q-binomial is $k(n-k)$.
- All coefficients are positive!
- In fact, the coefficients are unimodal. Sylvester, 1878

A few faces of the q-binomial coefficient

The q-binomial coefficient $\binom{n}{k}_q$

- satisfies a q-version of Pascal's rule, $\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$,

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- has a q-integral representation analogous to the beta function,
- counts the number of k-dimensional subspaces of $\mathbb{F}_q^n$.

A q-analog of Babbage's congruence

- Combinatorially, we again obtain: "q-Chu-Vandermonde"

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THM Clark 1995

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} \pmod{\Phi_n(q)^2}$$

- Note that $\Phi_n(1) = 1$ if n is not a prime power.

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- Similar results by Andrews (1999); e.g.:

$$\binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \qquad (\operatorname{mod} [p]_q^2)$$

A q-analog of Ljunggren's congruence

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THM S 2011/18
$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - (a-b)b\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}$$

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EG
$n=13$, $a=2$, $b=1$

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1+q+\ldots+q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132} \in \mathbb{Z}[q]$.

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- Note that $\frac{n^2-1}{24}$ is an integer if $(n,6)=1$.
- Ljunggren's classical congruence holds modulo p^{3+r} with r the p-adic valuation of Jacobsthal '52

$$(a-b)ab\binom{a}{b}.$$

A q-version of the Apéry numbers

- A symmetric q-analog of the Apéry numbers:

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2$$

This is an explicit form of a q-analog of Krattenthaler, Rivoal and Zudilin (2006).

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- The first few values are:

$$\begin{aligned} A(0) &= 1 & A_q(0) &= 1 \\ A(1) &= 5 & A_q(1) &= 1 + 3q + q^2 \\ A(2) &= 73 & A_q(2) &= 1 + 3q + 9q^2 + 14q^3 + 19q^4 + 14q^5 \\ & & &\quad + 9q^6 + 3q^7 + q^8 \\ A(3) &= 1445 & A_q(3) &= 1 + 3q + 9q^2 + 22q^3 + 43q^4 + 76q^5 \\ & & &\quad + 117q^6 + \ldots + 3q^{17} + q^{18} \end{aligned}$$

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2014/18

The q-analog of the Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

satisfies, for any $m \geqslant 0$,

$A_q(1) = 1 + 3q + q^2, \quad A(1) = 5$

$$A_q(mn) \equiv A_{q^{m^2}}(n) - \frac{m^2-1}{12}(q^m-1)^2 n^2 A_1(n) \pmod{\Phi_m(q)^3}.$$

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- Gorodetsky (2018) recently proved q-congruences implying the stronger congruences $A(p^r n) \equiv A(p^{r-1} n)$ modulo p^{3r}.

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- q-analog and congruences for Almkvist–Zudilin numbers? (classical supercongruences still open)

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

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Supercongruences for sporadic sequences
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Algebra & Number Theory, Vol. 8, Nr. 8, 2014, p. 1985-2008

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Proceedings of the American Mathematical Society, 2018