Properties and applications of Apéry-like numbers

Mathematics Colloquium
University of South Alabama

Armin Straub

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University of Illinois at Urbana–Champaign

$$A(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$$

$$1,5,73,1445,33001,819005,21460825,\ldots$$


Jon Borwein

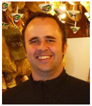

Dirk Nuyens


James Wan


Wadim Zudilin


Robert Osburn


Brundaban Sahu


Mathew Rogers

Rough outline

- Introducing Apéry-like numbers
  - they are integer solutions to certain three-term recurrences
  - are all of them known?
- Apéry-like numbers have interesting properties
  - connection to modular forms (uniform explanation?)
  - supercongruences (still open in several cases)
  - multivariate extensions (largely unexplored)
  - polynomial analogs (skipped today)
- Apéry-like numbers occur in interesting places
  - moments of planar random walks
  - series for $1/\pi$
  - positivity of rational functions (if time permits)
  - counting points on algebraic varieties (skipped today)
  - ...

The Riemann zeta function

- The **Riemann zeta function** is the analytic continuation of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

- Its zeros and values are fundamental, yet mysterious to this day.

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THM Euler 1734:

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \ldots, \quad \zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n} B_{2n}}{2(2n)!}$$

CONJ The values $\zeta(3), \zeta(5), \zeta(7), \ldots$ are all transcendental.

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THM Apéry '78: $\zeta(3)$ is irrational.

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2+17n+5)A(n) - n^3 A(n-1).$$

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THM (Apéry '78) $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q **Beukers, Zagier** Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

Zagier's search and Apéry-like numbers

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Q (Beukers, Zagier): Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

- Essentially, only 14 tuples (a, b, c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions
  - 6 sporadic solutions

- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}$ (Cooper)

The six sporadic Apéry-like numbers

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

- The Apéry numbers $A(n)$ satisfy $\qquad 1, 5, 73, 1145, \ldots$

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad q - 12q^2 + 66q^3 + O(q^4)$ $\qquad q = e^{2\pi i\tau}$

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- The **Dedekind eta function**

$$\eta(\tau) = q^{1/24} \prod_{n\geqslant 1} (1 - q^n)$$

is a modular form of weight $1/2$ and transforms as

$$\eta(\tau+1) = e^{\pi i/12}\eta(\tau), \qquad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau).$$

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FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

Supercongruences for Apéry numbers

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THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$
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EG For primes p, simple combinatorics proves the congruence
$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$
For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,
$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers.

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
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$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open!! modulo p^2 Amdeberhan '14
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	Osburn–Sahu–S '14
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	open

Non-super congruences are abundant

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(C)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T: X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{“points of period } n\text{”}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

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- $a(n) = \operatorname{ct} \Lambda(x)^n$ van Straten–Samol '09

  if origin is only interior pt of the Newton polyhedron of $\Lambda(x) \in \mathbb{Z}_p[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$

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- If $a(1) = 1$, then (C) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

Cooper's sporadic sequences

- Cooper's search for integral solutions to

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$$

revealed three additional sporadic solutions: s_{10} and supercongruence known

$$s_7(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \qquad s_{10}(n) = \sum_{k=0}^{n} \binom{n}{k}^4$$

$$s_{18}(n) = \sum_{k=0}^{[n/3]} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n} \right]$$

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CONJ Cooper 2012

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THM Osburn-Sahu-S 2014

$$s_7(mp^r) \equiv s_7(mp^{r-1}) \pmod{p^{3r}} \qquad p \geqslant 5$$

$$s_{18}(mp^r) \equiv s_{18}(mp^{r-1}) \pmod{p^{2r}}$$

Diagonals

- Given a series

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n, \ldots, n)$.

EG The diagonal coefficients of

$$\frac{1}{1 - x - y}$$

are the central binomial coefficients $\binom{2n}{n}$.

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EG The diagonal coefficients of

$$\frac{1}{1 - x - y} = \sum_{n=0}^{\infty} (x + y)^n$$

are the central binomial coefficients $\binom{2n}{n}$.
For comparison, their univariate generating function is

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1 - 4x}}.$$

THM
S 2014

The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}.$$

THM S 2014 The Apéry numbers are the diagonal coefficients of

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- Univariate generating function:

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}}\, {}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix}\middle| -\frac{1024x}{(1-x+z)^4}\right),$$

where $z=\sqrt{1-34x+x^2}$.

Apéry numbers as diagonals

THM
S 2014

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where $z = \sqrt{1 - 34x + x^2}$.

- Well-developed theory of multivariate asymptotics   e.g., Pemantle–Wilson

Apéry numbers as diagonals

THM S 2014 The Apéry numbers are the diagonal coefficients of

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  where $z=\sqrt{1-34x+x^2}$.
- Well-developed theory of multivariate asymptotics e.g., Pemantle–Wilson
- Such diagonals are algebraic modulo p^r. Furstenberg, Deligne '67, '84

  Automatically leads to congruences such as

$$A(n) \equiv \begin{cases} 1 & \pmod 8, \quad \text{if } n \text{ even,} \\ 5 & \pmod 8, \quad \text{if } n \text{ odd.}\end{cases}$$

Chowla–Cowles–Cowles '80
Rowland–Yassawi '13

THM
S 2014

Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

THM S 2014

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- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- $\displaystyle\sum_{n\geqslant 0} a(n)x^n = F(x) \quad\Longrightarrow\quad \sum_{n\geqslant 0} a(pn)x^{pn} = \frac{1}{p}\sum_{k=0}^{p-1} F(\zeta_p^k x)$ $\qquad \zeta_p = e^{2\pi i/p}$
- Hence, both $A(\boldsymbol{n}p^r)$ and $A(\boldsymbol{n}p^{r-1})$ have rational generating function.
  The proof, however, relies on an explicit binomial sum for the coefficients.

CONJ S 2014 The coefficients $Z(\boldsymbol{n})$ of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^4} Z(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

satisfy, for $p \geqslant 5$, the multivariate supercongruences

$$Z(\boldsymbol{n}p^r) \equiv Z(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- Here, the diagonal coefficients are the **Almkvist–Zudilin numbers**

$$Z(n) = \sum_{k=0}^{n} (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3},$$

for which the univariate congruences are still open.

Short random walks

joint work with:

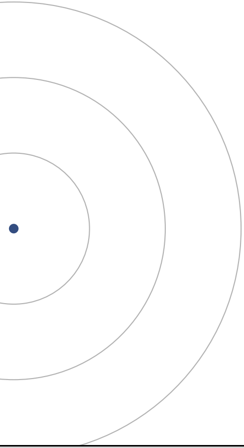
Jon Borwein
U. Newcastle, AU

Dirk Nuyens
K.U.Leuven, BE

James Wan
SUTD, SG

Wadim Zudilin
U. Newcastle, AU

n steps in the plane
(length 1, random direction)



What is the distance traveled in n steps?

$p_n(x)$	probability density
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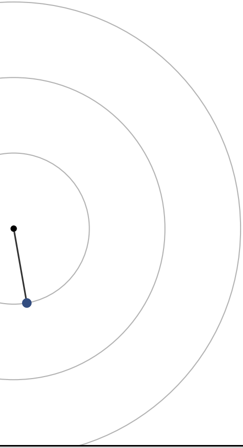

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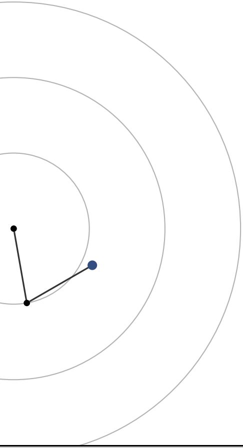

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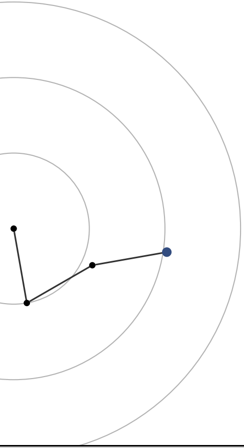

What is the distance traveled in n steps?

$p_n(x)$	probability density
$W_n(s)$	sth moment

Random walks in the plane

n steps in the plane
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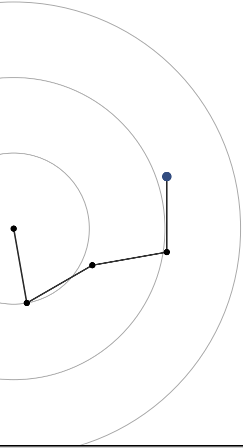

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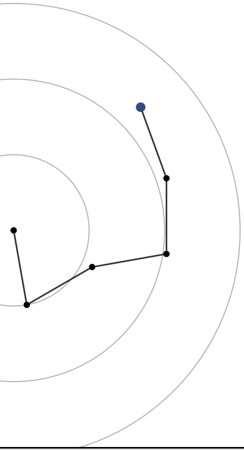

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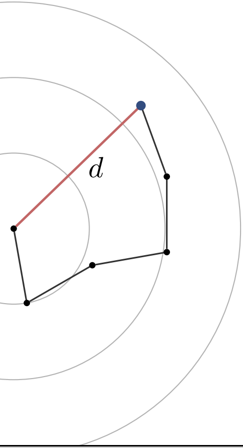

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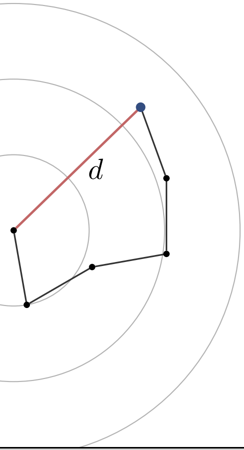

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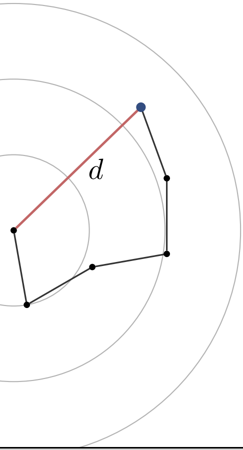

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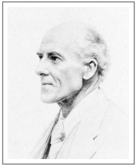

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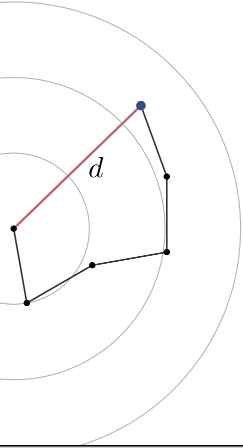

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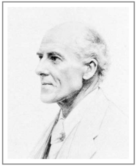

THM Rayleigh, 1905

$$p_n(x) \approx \frac{2x}{n} e^{-x^2/n} \qquad \text{for large } n$$

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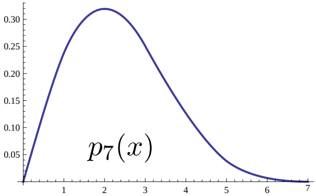

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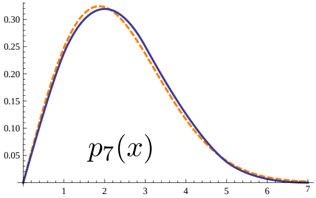

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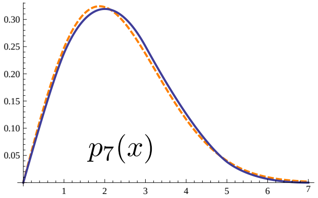

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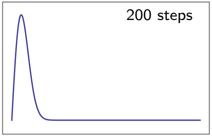
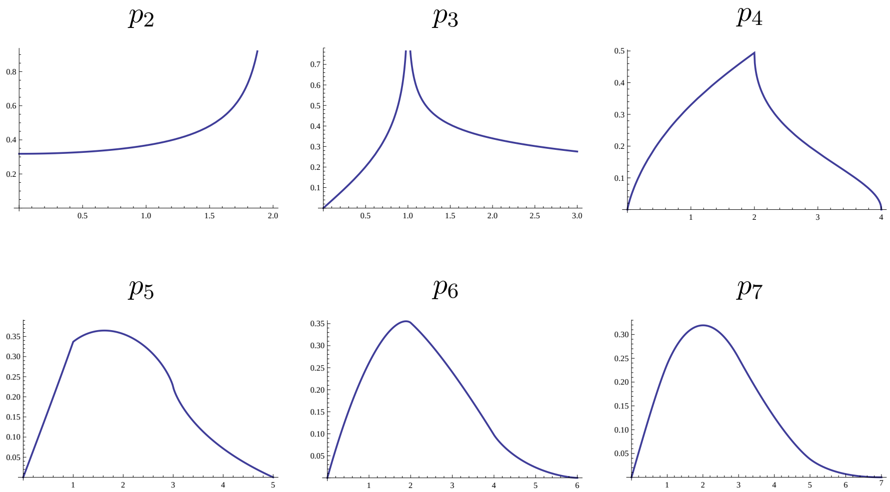
> *The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point!*
>
> **Karl Pearson**, *1905*



Densities of short walks



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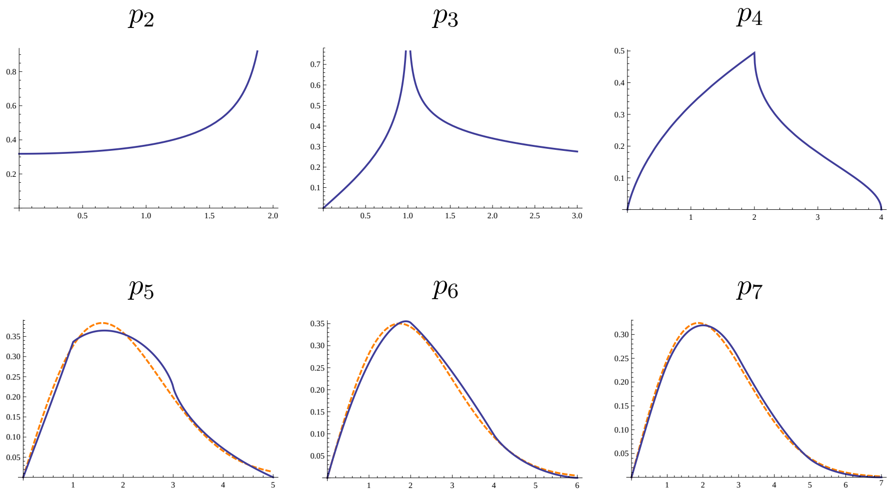

Classical results on the densities

$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \mathrm{Re}\left(\frac{\sqrt{x}}{\pi^2}\,K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right)\right)$$ G. J. Bennett 1905

$$p_4(x) = \,??$$

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$$p_n(x) = \int_0^\infty xtJ_0(xt)J_0^n(t)\,\mathrm{d}t$$ J. C. Kluyver 1906

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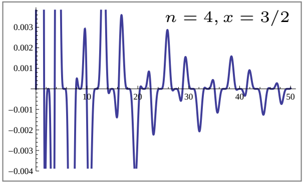

The average distance traveled in two steps

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x \mathrm{d}y = \ ?$$

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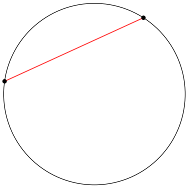
- This is the average length of a random arc on a unit circle.



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$$\begin{aligned} W_3(1) &\approx 1.57459723755189365749 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

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- Hard to evaluate numerically to high precision.
  Monte-Carlo integration gives approximations with an asymptotic error of $O(1/\sqrt{N})$ where N is the number of sample points.

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n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
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The even moments

n	$s=0$	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$	Sloane's
2	1	2	6	20	70	252	A000984
3	1	3	15	93	639	4653	A002893
4	1	4	28	256	2716	31504	A002895
5	1	5	45	545	7885	127905	A169714
6	1	6	66	996	18306	384156	A169715

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

Apéry-like

$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}$$

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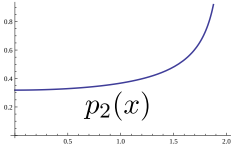
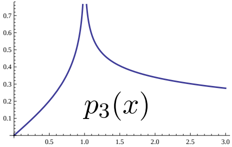
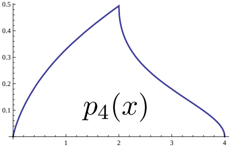
THM Borwein-Nuyens-S-Wan, 2010

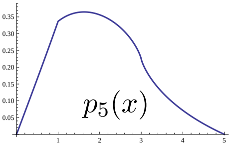
$$W_3(1) = \frac{3}{16} \frac{2^{1/3}}{\pi^4} \Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4} \frac{2^{2/3}}{\pi^4} \Gamma^6\left(\frac{2}{3}\right)$$

Densities of random walks









$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$

easy

$$p_3(x) = \frac{2\sqrt{3}}{\pi}\,\frac{x}{(3+x^2)}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\right|\frac{x^2\left(9-x^2\right)^2}{\left(3+x^2\right)^3}\right)$$

classical
with a spin

$$p_4(x) = \frac{2}{\pi^2}\,\frac{\sqrt{16-x^2}}{x}\ \mathrm{Re}\ {}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{5}{6},\frac{7}{6}\end{matrix}\right|\frac{\left(16-x^2\right)^3}{108x^4}\right)$$

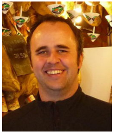
new
BSWZ 2011

$$p_5'(0) = \frac{\sqrt{5}}{40\pi^4}\,\Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

Ramanujan-type series for $1/\pi$

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots$$

Based on joint work with:



Mathew Rogers
(University of Montreal)

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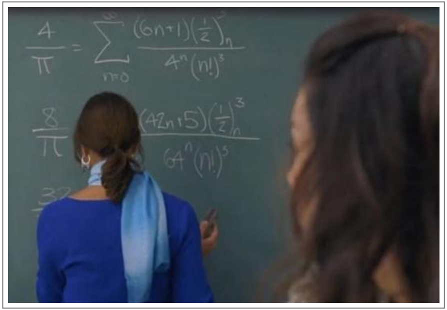
$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{8}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$



- Starred in High School Musical, a 2006 Disney production

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

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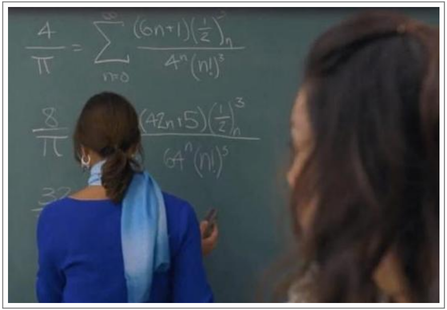

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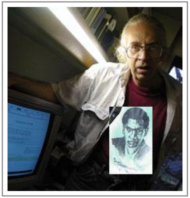
Another one of Ramanujan's series

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1103 + 26390n}{396^{4n}}$$



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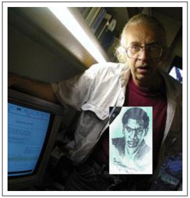

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- First proof of all of Ramanujan's 17 series for $1/\pi$ by Borwein brothers

Jonathan M. Borwein and Peter B. Borwein
Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity
Wiley, 1987

- Sato observed that series for $\frac{1}{\pi}$ can be built from Apéry-like numbers:

EG **Chan-Chan-Liu 2003**

For the Domb numbers $D(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$,

$$\frac{8}{\sqrt{3}\pi} = \sum_{n=0}^{\infty} D(n) \frac{5n+1}{2^{6n}}.$$

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- Sun offered a \$520 bounty for a proof the following series:

THM
Rogers-S 2012

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n+233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

A brief guide to proving series for $1/\pi$

- Suppose we have a sequence a_n with **modular parametrization**

$$\sum_{n=0}^{\infty} a_n \underbrace{x(\tau)^n}_{\text{modular function}} = \underbrace{f(\tau)}_{\text{modular form}} .$$

- Then:

$$\sum_{n=0}^{\infty} a_n (A + Bn) x(\tau)^n = A f(\tau) + B \frac{x(\tau)}{x'(\tau)} f'(\tau)$$

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3} (42n+5) \frac{1}{2^{6n}} = \frac{16}{\pi}$$

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FACT

- For $\tau \in \mathbb{Q}(\sqrt{-d})$, $x(\tau)$ is an algebraic number.
- $f'(\tau)$ is a **quasimodular** form.
- Prototypical $E_2(\tau)$ satisfies $\tau^{-2} E_2\left(-\frac{1}{\tau}\right) - E_2(\tau) = \frac{6}{\pi i \tau}$.

- These are the main ingredients for series for $1/\pi$. Mix and stir.

Positivity of rational functions

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}$$

Based on joint work with:

Wadim Zudilin
(University of Newcastle)

Positivity of rational functions

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1,\ldots,n_d \geqslant 0} a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1,\ldots,n_d} > 0$ for all indices.

EG The following rational functions are positive.

$$S(x,y,z) = \frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}$$

Szegő '33
Kaluza '33
Askey–Gasper '72
S '08

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Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

- Both functions are on the boundary of positivity.

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EG The following rational functions are positive.

$$S(x, y, z) = \frac{1}{1 - (x + y + z) + \frac{3}{4}(xy + yz + zx)}$$

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- Both functions are on the boundary of positivity.
- The diagonal coefficients of A are the **Franel numbers** $\sum_{k=0}^{n} \binom{n}{k}^3$.

CONJ Kauers-Zeilberger 2008

The following rational function is positive:

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

- Would imply conjectured positivity of Lewy–Askey rational function

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Recent proof of non-negativity by Scott and Sokal, 2013

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PROP S-Zudilin 2013

The Kauers–Zeilberger function has diagonal coefficients

$$d_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2.$$

Positivity of rational functions

- Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with p a symmetric polynomial, linear in each variable.

Q Under what condition(s) is the positivity of F implied by the positivity of its diagonal?

EG
- $\frac{1}{1-(x+y)}$ is positive.
- $\frac{1}{1+x+y}$ has positive diagonal but is not positive.

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THM S-Zudilin 2013

$$F(x,y) = \frac{1}{1 + c_1(x+y) + c_2 xy} \quad \text{is positive}$$

$\iff$ diagonal of F and $F|_{y=0}$ are positive

Summary and some open problems

- Apéry-like numbers are integer solutions to certain three-term recurrences
  - is the experimental list complete?
  - higher-order analogs, Calabi–Yau DEs
- Apéry-like numbers have interesting properties
  - modular parametrization; uniform explanation?
  - supercongruences; still open in several cases
  - do they all have natural multivariate extensions?
  - polynomial analogs
- Apéry-like numbers occur in interesting places
  - moments of planar random walks
  - series for $1/\pi$
  - positivity of rational functions
  - counting points on algebraic varieties
  - ...

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

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Multivariate Apéry numbers and supercongruences of rational functions
Algebra & Number Theory, Vol. 8, Nr. 8, 2014, p. 1985-2008

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Supercongruences for sporadic sequences
to appear in Proceedings of the Edinburgh Mathematical Society, 2014

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to appear in Journal of Approximation Theory (special issue dedicated to Richard Askey), 2014

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Canadian Journal of Mathematics, Vol. 64, Nr. 5, 2012, p. 961-990