Supercongruences for Apéry-like numbers

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$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$1, 5, 73, 1445, 33001, 819005, \ldots$

Includes joint work with:

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2 + 17n + 5)u_n - n^3 u_{n-1}.$$

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THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q Beukers, Zagier: Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

Zagier's search and Apéry-like numbers

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Q (Beukers, Zagier): Are there other tuples (a,b,c) for which the solution defined by $u_{-1}=0$, $u_0=1$ is integral?

- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions
  - 6 sporadic solutions
- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2+an+b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$ (Cooper)

- Hypergeometric and Legendrian solutions have generating functions

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\ 1,1\end{matrix}\middle| 4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\begin{matrix}\alpha,1-\alpha\\ 1\end{matrix}\middle| \frac{-C_\alpha z}{1-C_\alpha z}\right)^2, $$

  with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4 \cdot 3^3$.

- The six sporadic solutions are:

(a,b,c)	$A(n)$
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$
$(10,4,64)$	$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$
$(12,4,16)$	$\sum_k \binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1)$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$

Modularity of Apéry-like numbers

- The **Apéry numbers** $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} \right)^{12n}}_{\text{modular function}}.$$

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FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- As a consequence,

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}} \, {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1\end{matrix} \middle| -\frac{1024x}{(1-x+z)^4}\right),$$

where $z = \sqrt{1 - 34x + x^2}$.

Supercongruences for Apéry numbers

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THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

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EG

Simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1+1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers. Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$	open!! modulo p^2 Amdeberhan '14
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$	Osburn–Sahu–S '14
$(10,4,64)$	$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$	Osburn–Sahu '11
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$(17,5,1)$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$	Beukers, Coster '87-'88

Non-super congruences are abundant

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(C)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{“points of period } n\text{”}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

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- $a(n) = \operatorname{ct} \Lambda(x)^n$ van Straten–Samol '09

  if origin is only interior pt of the Newton polyhedron of $\Lambda(x) \in \mathbb{Z}_p[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$

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- $a(n) = \sum_{d|n} \chi(d) d^k$ satisfies (C) modulo p^{kr}. — Exercise!

Cooper's sporadic sequences

- Cooper's search for integral solutions to

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$$

revealed three additional sporadic solutions: s_{10} studied before

$$s_7(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \qquad s_{10}(n) = \sum_{k=0}^{n} \binom{n}{k}^4$$

$$s_{18}(n) = \sum_{k=0}^{[n/3]} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n} \right]$$

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CONJ Cooper 2012

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THM Osburn-Sahu-S 2014

$$s_7(mp^r) \equiv s_7(mp^{r-1}) \pmod{p^{3r}} \qquad p \geqslant 5$$

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First generalization: multivariate supercongruences

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Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the **diagonal coefficients**.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

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- Both $A(\boldsymbol{n} p^r)$ and $A(\boldsymbol{n} p^{r-1})$ have rational generating function.
  The proof, however, relies on an explicit binomial sum for the coefficients.

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THM S 2013

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- Both $A(\boldsymbol{n}p^r)$ and $A(\boldsymbol{n}p^{r-1})$ have rational generating function.
  The proof, however, relies on an explicit binomial sum for the coefficients.
- Such diagonals are algebraic modulo p^r. Furstenberg, Deligne '67, '84
  Automatically (pun intended) leads to congruences such as

$$A(n) \equiv \begin{cases} 1 \pmod 8, & \text{if } n \text{ even}, \\ 5 \pmod 8, & \text{if } n \text{ odd}. \end{cases}$$

Chowla–Cowles–Cowles '80
Rowland–Yassawi '13

A simple conjectural tip of an iceberg

CONJ S 2013 The coefficients $F(\boldsymbol{n})$ of

$$\frac{1}{1-(x_1+x_2+x_3)+4x_1x_2x_3} = \sum_{\boldsymbol{n}\in\mathbb{Z}^3_{\geqslant 0}} F(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

satisfy, for $p \geqslant 5$, the multivariate supercongruences

$$F(\boldsymbol{n}p^r) \equiv F(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- Here, the diagonal coefficients are the **Franel numbers**

$$F(n) = \sum_{k=0}^{n} \binom{n}{k}^3.$$

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- Here, the diagonal coefficients are the **Franel numbers**

$$F(n) = \sum_{k=0}^{n} \binom{n}{k}^3.$$

- The Franel numbers also appear as the diagonal coefficients of

$$\frac{1}{(1-x_1)(1-x_2)(1-x_3)-x_1x_2x_3},$$

for which we can prove the above multivariate supercongruences.

Second generalization: q-supercongruences

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The q-analog of the Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{k^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

satisfy

$A_q(1) = 1 + q + 2q^2 + q^3, \quad A(1) = 5$

$$A_q(pn) - A_{q^{p^2}}(n) \equiv -\frac{p^2-1}{12}(q^p-1)^2 f(n) \pmod{[p]_q^3}.$$

Second generalization: q-supercongruences

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satisfy $\qquad A_q(1) = 1 + q + 2q^2 + q^3, \quad A(1) = 5$

$$A_q(pn) - A_{q^{p^2}}(n) \equiv -\frac{p^2-1}{12}(q^p-1)^2 f(n) \pmod{[p]_q^3}.$$

- The numbers $f(n)$ can be expressed as $\qquad 0, 5, 292, 13005, 528016, \ldots$

$$f(n) = \sum_{k=0}^{n} g(n,k) \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad g(n,k) = k(2n-k) + \frac{k^4}{(n+k)^2}.$$

- Similar congruences for other Apéry-like numbers?

Some of many open problems

- Supercongruences for all Apéry-like numbers
  - proof for all of them
  - uniform explanation
  - multivariable extensions
- Apéry-like numbers as diagonals
  - find minimal rational functions
  - extend supercongruences
  - any structure?
- Many further questions remain.
  - is the known list complete?
  - higher-order analogs, Calabi–Yau DEs
  - reason for modularity
  - modular supercongruences — Beukers '87, Ahlgren–Ono '00

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}, \qquad \sum_{n=1}^{\infty} a(n) q^n = \eta^4(2\tau)\eta^4(4\tau)$$

  - q-analogs
  - ...

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions
Preprint, 2014

R. Osburn, B. Sahu, A. Straub
Supercongruences for sporadic sequences
to appear in Proceedings of the Edinburgh Mathematical Society, 2014

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
to appear in Journal of Approximation Theory (special issue dedicated to Richard Askey), 2014

Fuller version of main result

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Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}_{>0}^\ell$ with $d = \lambda_1 + \ldots + \lambda_\ell$, and set $s(j) = \lambda_1 + \ldots + \lambda_{j-1}$. Define $A_\lambda(\boldsymbol{n})$ by

$$\left(\prod_{j=1}^{\ell} \left[1 - \sum_{r=1}^{\lambda_j} x_{s(j)+r} \right] - x_1 x_2 \cdots x_d \right)^{-1} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^d} A_\lambda(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- If $\ell \geqslant 2$, then, for all primes p and integers $r \geqslant 1$,

$$A_\lambda(\boldsymbol{n}p^r) \equiv A_\lambda(\boldsymbol{n}p^{r-1}) \pmod{p^{2r}}.$$

- If $\ell \geqslant 2$ and $\max(\lambda_1, \ldots, \lambda_\ell) \leqslant 2$, then, for primes $p \geqslant 5$ and integers $r \geqslant 1$,

$$A_\lambda(\boldsymbol{n}p^r) \equiv A_\lambda(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$