SHORT WALK ADVENTURES

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To the memory of Jon Borwein, who convinced us that a short walk can be adventurous

Abstract. We review recent development of short uniform random walks, with a focus on its connection to (zeta) Mahler measures and modular parametrisation of the density functions. Furthermore, we extend available "probabilistic" techniques to cover a variation of random walks and reduce some three-variable Mahler measures, which are conjectured to evaluate in terms of L-values of modular forms, to hypergeometric form.

0. Introduction

At some stages of our careers we were approached by Jon Borwein to collaborate on a theme that sounded rather off topic to us, who had interests in number theory, combinatorics and related special functions. Somewhat unexpectedly, the theme has become a remarkable research project with several outcomes (including [9, 10, 11], to list a few), a project which we continue to enjoy after the sudden loss of Jon... This note serves as a summary to our recent discoveries that certain "probabilistic" techniques apply usefully to tackling difficult problems on the border of analysis, number theory and differential equations; in particular, in evaluating multi-variable Mahler measures. Our principal novelties are given in Theorems 1–3; these include hypergeometric reduction of the Mahler measures of the three-variable polynomials

$$1 + x_1 + x_2 + x_3 + x_2x_3 \quad \text{and} \quad (1 + x_1)^2 + x_2 + x_3,$$

as well as the (hypergeometric) factorisation of a related differential operator for the Apéry-like sequence

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}^2, \quad \text{where } n = 0, 1, 2, \dots .$$

Echoing Jon's "a short walk can be beautiful" [8], we add that "a short walk can be adventurous."

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1. Uniform random walks

An N-step uniform random walk is a planar walk that starts at the origin and consists of N steps of length 1 each taken into a uniformly random direction. Let X_N be the distance to the origin after these N steps. The s-th moments $W_N(s)$ of X_N can be computed [11] via the formula

$$\begin{aligned} W_N(s) &= \int\cdots\int_{[0,1]^N} |e^{2\pi i\theta_1}+\cdots+e^{2\pi i\theta_N}|^s \,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_N \\ &= \int\cdots\int_{[0,1]^{N-1}} |1+e^{2\pi i\theta_1}+\cdots+e^{2\pi i\theta_{N-1}}|^s \,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{N-1}, \end{aligned}$$

and are related to the (probability) density function $p_N(x)$ of X_N via

$$W_N(s) = \int_0^\infty x^s p_N(x)\,\mathrm{d}x = \int_0^N x^s p_N(x)\,\mathrm{d}x.$$

That is, $p_N(x)$ can then be obtained as the inverse Mellin transform of $W_N(s-1)$. Finally, note that the even moments $W_3(2n)$ and $W_4(2n)$ (which are, clearly, positive integers) can be identified with the odd moments of $I_0(t)K_0(t)^2$ and $I_0(t)K_0(t)^3$, respectively, where $I_0(t)$ and $K_0(t)$ denote the modified Bessel functions of the first and second kind. Namely, for $n=1,2,\dots$ we have [6]

$$W_3(2n) = \frac{3^{2n+3/2}}{\pi\, 2^{2n}\, n!^2}\int_0^\infty t^{2n+1} I_0(t)K_0(t)^2\,\mathrm{d}t$$

and

$$W_4(2n) = \frac{4^{2n+2}}{\pi^2\, n!^2}\int_0^\infty t^{2n+1} I_0(t)K_0(t)^3\,\mathrm{d}t.$$

2. Zeta Mahler measures

For a non-zero Laurent polynomial $P(x_1,\dots,x_N)\in\mathbb{C}[x_1^{\pm1},\dots,x_N^{\pm1}]$, its zeta Mahler measure [3] is defined by

$$Z(P;s) = \int\cdots\int_{[0,1]^N} |P(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_N})|^s\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_N,$$

and its logarithmic Mahler measure is

$$\mathrm{m}(P) = \left.\frac{\mathrm{d}Z(P;s)}{\mathrm{d}s}\right|_{s=0} = \int\cdots\int_{[0,1]^N} \log|P(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_N})|\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_N.$$

A straightforward comparison of the two definitions reveals that

$$W_N(s) = Z(x_1+\cdots+x_N;s) = Z(1+x_1+\cdots+x_{N-1};s)$$

and

$$W_N'(0) = \mathrm{m}(x_1+\cdots+x_N) = \mathrm{m}(1+x_1+\cdots+x_{N-1}) = \int_0^N p_N(x)\log x\,\mathrm{d}x, \quad (1)$$

where the derivative is with respect to s. The latter Mahler measures are known as linear Mahler measures. The evaluations

$$W_2'(0) = 0, \quad W_3'(0) = L'(\chi_{-3}; -1) = \frac{3\sqrt{3}}{4\pi}\, L(\chi_{-3}; 2), \quad W_4'(0) = -14\zeta'(-2) = \frac{7\zeta(3)}{2\pi^2},$$

are known [24], while the following conjectural evaluations, due to Rodriguez-Villegas [13] and verified to several hundred digits [5], remain open:

$$W_5'(0) \stackrel{?}{=} -L'(f_3; -1) = 6\left(\frac{\sqrt{15}}{2\pi}\right)^5 L(f_3; 4),$$
$$W_6'(0) \stackrel{?}{=} -8L'(f_4; -1) = 3\left(\frac{\sqrt{6}}{\pi}\right)^6 L(f_4; 5),$$

where

$$f_3(\tau) = \eta(\tau)^3\eta(15\tau)^3 + \eta(3\tau)^3\eta(5\tau)^3 \quad \text{and} \quad f_4(\tau) = \eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$$

are cusp eigenforms of weight 3 and 4, respectively. Here and in what follows, Dedekind's eta function

$$\eta(\tau) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24}, \quad \text{where } q = e^{2\pi i\tau},$$

serves as a principal constructor of modular forms and functions. No similar formulae are known for $W_N'(0)$ when $N \geq 7$, though the story continues at a different level — see [14, 30, 31] for details.

3. Generic two-step random walks

Let X_1 and X_2 be two (sufficiently nice, independent) random variables on $[0, \infty)$ with probability density $p_1(x)$ and $p_2(x)$, respectively, and let θ_1 and θ_2 be uniformly distributed on $[0, 1]$. Then $X = e^{2\pi i\theta_1}X_1 + e^{2\pi i\theta_2}X_2$ describes a two-step random walk in the plane with a first step of length X_1 and a second step of length X_2. As in [10, eq. (3-3)], an application of the cosine rule shows that the s-th moment of $|X|$ is

$$W(s) = \mathsf{E}(|X|^s) = \int_0^\infty\int_0^\infty g_s(x, y)p_1(x)p_2(y)\,\mathrm{d}x\,\mathrm{d}y,$$

where

$$g_s(x, y) = \frac{1}{\pi}\int_0^\pi (x^2 + y^2 + 2xy\cos\theta)^{s/2}\,\mathrm{d}\theta.$$

Observe that

$$\left.\frac{\mathrm{d}g_s(x, y)}{\mathrm{d}s}\right|_{s=0} = \frac{1}{\pi}\int_0^\pi \log\sqrt{x^2 + y^2 + 2xy\cos\theta}\,\mathrm{d}\theta = \max\{\log|x|, \log|y|\},$$

so that, in particular,

Lemma 1. *We have*

$$W'(0) = \mathsf{E}(\log|X|) = \int_0^\infty\int_0^\infty p_1(x)p_2(y)\max\{\log x, \log y\}\,\mathrm{d}y\,\mathrm{d}x.$$

Alternative equivalent expressions, that will be useful in what follows, include

$$\begin{aligned}\mathsf{E}(\log|X|) &= \int_0^\infty\int_0^x p_1(x)p_2(y)\log x\,\mathrm{d}y\,\mathrm{d}x + \int_0^\infty\int_x^\infty p_1(x)p_2(y)\log y\,\mathrm{d}y\,\mathrm{d}x \\ &= \mathsf{E}(\log X_1) + \int_0^\infty\int_x^\infty p_1(x)p_2(y)(\log y - \log x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \mathsf{E}(\log X_2) + \int_0^\infty\int_0^x p_1(x)p_2(y)(\log x - \log y)\,\mathrm{d}y\,\mathrm{d}x. \qquad (2)\end{aligned}$$

4. Linear Mahler measures

Let N, M be integers such that $N > M > 0$. By decomposing an N-step random walk into two walks with $N-M$ and M steps, and applying Lemma 1 in the form (2), we find that

$$W_N'(0) = W_M'(0) + \int_0^{N-M} p_{N-M}(x)\left(\int_0^x p_M(y)(\log x - \log y)\,\mathrm{d}y\right)\mathrm{d}x.$$

This formula, together with known formulae for the densities [11], like $p_1(x) = \delta(x-1)$ (the Dirac delta function) and $p_2(x) = 2/(\pi\sqrt{4-x^2})$ for $0 < x < 2$, allows one to produce new expressions for linear Mahler measures. Indeed, taking $M = 1$ we get

$$W_N'(0) = \int_1^{N-1} p_{N-1}(x)\log x\,\mathrm{d}x \qquad (3)$$

(which can be also derived using Jensen's formula), while $M = 2$ results in

$$W_N'(0) = \int_2^{N-2} p_{N-2}(x)\log x\,\mathrm{d}x + \frac{1}{\pi}\int_0^2 p_{N-2}(x)x\cdot {}_3F_2\left(\begin{matrix}\frac12, \frac12, \frac12 \\ \frac32, \frac32\end{matrix}\,\middle|\,\frac{x^2}{4}\right)\mathrm{d}x \qquad (4)$$

(see also [20, eq. (2.1)]). Here, and in what follows, the hypergeometric notation

$${}_mF_{m-1}\left(\begin{matrix}a_1, a_2, \dots, a_m \\ b_2, \dots, b_m\end{matrix}\,\middle|\,z\right) = \sum_{n=0}^\infty \frac{(a_1)_n(a_2)_n\cdots(a_m)_n}{(b_2)_n\cdots(b_m)_n}\frac{z^n}{n!}$$

is used, where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+n-1), & \text{for } n \geq 1, \\ 1, & \text{for } n = 0,\end{cases}$$

denotes the Pochhammer symbol (the rising factorial). Note that we deduce (4) from

$$\int_0^x p_2(y)(\log x - \log y)\,\mathrm{d}y = \frac{x}{\pi}\cdot {}_3F_2\left(\begin{matrix}\frac12, \frac12, \frac12 \\ \frac32, \frac32\end{matrix}\,\middle|\,\frac{x^2}{4}\right),$$

which is valid if $0 \leq x \leq 2$.

Equations (3) and (4) and the formula

$$p_4(x) = \frac{2\sqrt{16-x^2}}{\pi^2 x}\,\mathrm{Re}\,{}_3F_2\left(\begin{matrix}\frac12, \frac12, \frac12 \\ \frac56, \frac76\end{matrix}\,\middle|\,\frac{(16-x^2)^3}{108x^4}\right)$$

obtained in [11, Theorem 4.9], provide the formulae

$$W_5'(0) = \frac{7\zeta(3)}{2\pi^2} - \frac{1}{\pi^2}\int_0^1 \sqrt{16-x^2}\,\mathrm{Re}\,{}_3F_2\left(\begin{matrix}\frac12,\frac12,\frac12\\ \frac56,\frac76\end{matrix}\,\middle|\,\frac{(16-x^2)^3}{108x^4}\right)\mathrm{d}(\log^2 x)$$

and

$$W_6'(0) = \frac{7\zeta(3)}{2\pi^2} - \frac{1}{\pi^2}\int_0^2 \sqrt{16-x^2}\,\mathrm{Re}\,{}_3F_2\left(\begin{matrix}\frac12,\frac12,\frac12\\ \frac56,\frac76\end{matrix}\,\middle|\,\frac{(16-x^2)^3}{108x^4}\right)\mathrm{d}(\log^2 x)$$
$$+\frac{2}{\pi^3}\int_0^2 \sqrt{16-x^2}\,\mathrm{Re}\,{}_3F_2\left(\begin{matrix}\frac12,\frac12,\frac12\\ \frac56,\frac76\end{matrix}\,\middle|\,\frac{(16-x^2)^3}{108x^4}\right)\cdot {}_3F_2\left(\begin{matrix}\frac12,\frac12,\frac12\\ \frac32,\frac32\end{matrix}\,\middle|\,\frac{x^2}{4}\right)\mathrm{d}x.$$

These *single* integrals can be used to numerically confirm the conjectural evaluations of $W_5'(0)$ and $W_6'(0)$.

A similar application of Lemma 1, upon decomposing a 6-step walk into two walks with 3 steps, yields the alternative reduction

$$W_6'(0) = 2\int_0^3 p_3(x)\log x\left(\int_0^x p_3(y)\,\mathrm{d}y\right)\mathrm{d}x, \tag{5}$$

where [11]

$$p_3(x) = \frac{2\sqrt{3}x}{\pi(3+x^2)}\cdot{}_2F_1\left(\begin{matrix}\frac13,\frac23\\ 1\end{matrix}\,\middle|\,\frac{x^2(9-x^2)^2}{(3+x^2)^3}\right).$$

We discuss this formula further in Section 5.

Finally, we mention that equation (3) and a modular parametrisation of $p_4(x)$ (which we indicate in Section 6) were independently cast in [23] to produce a double L-value expression for $W_5'(0)$.

5. Modular parametrisation of $p_3(x)$ and related formulae

Note that formula (5) can be written as

$$W_6'(0) = \int_0^3 \log x\,\mathrm{d}(P_3(x)^2) = \log 3 - \int_0^3 P_3(x)^2\,\frac{\mathrm{d}x}{x},$$

featuring the cumulative density function

$$P_3(x) = \int_0^x p_3(y)\,\mathrm{d}y.$$

The related modular parametrisation of $p_3(x)$ is given by

$$x = x(\tau) = 3\frac{\eta(\tau)^2\eta(6\tau)^4}{\eta(2\tau)^4\eta(3\tau)^2}\colon (i\infty,0)\to(0,3),$$

so that

$$p_3(x) = \frac{2\sqrt{3}}{\pi}\,\frac{\eta(2\tau)^2\eta(6\tau)^2}{\eta(\tau)\eta(3\tau)},\qquad \mathrm{d}x = 3\pi i\,\frac{\eta(\tau)^6\eta(3\tau)^2\eta(6\tau)^2}{\eta(2\tau)^6}\,\mathrm{d}\tau$$

and

$$P_3(x) = 6i\sqrt{3}\int_{i\infty}^{\tau}\frac{\eta(\tau)^5\eta(3\tau)\eta(6\tau)^4}{\eta(2\tau)^4}\,\mathrm{d}\tau$$

is the anti-derivative of a weight 3 holomorphic Eisenstein series

$$\frac{\eta(\tau)^5\eta(3\tau)\eta(6\tau)^4}{\eta(2\tau)^4} = E_{3,\chi_{-3}}(\tau) - 8E_{3,\chi_{-3}}(2\tau),$$

where

$$E_{3,\chi_{-3}}(\tau) = \frac{\eta(3\tau)^9}{\eta(\tau)^3} = \sum_{m,n=1}^{\infty}\left(\frac{-3}{m}\right)n^2q^{mn}, \quad \chi_{-3}(m) = \left(\frac{-3}{m}\right) = \frac{e^{2\pi im/3}-e^{-2\pi im/3}}{i\sqrt{3}}.$$

Though the anti-derivative $P_3(x)$,

$$\begin{aligned} P_3(x) &= \frac{3\sqrt{3}}{\pi}\Bigg(\sum_{m,n=1}^{\infty}\left(\frac{-3}{m}\right)\frac{n}{m}\,q^{mn} - 4\sum_{m,n=1}^{\infty}\left(\frac{-3}{m}\right)\frac{n}{m}\,q^{2mn}\Bigg) \\ &= \frac{9i}{\pi}\,\log\prod_{n=1}^{\infty}\left(\frac{(1-e^{2\pi i/3}q^{2n})^4(1-e^{-2\pi i/3}q^n)}{(1-e^{-2\pi i/3}q^{2n})^4(1-e^{2\pi i/3}q^n)}\right)^n, \end{aligned}$$

is not considered to be sufficiently "natural", it shows up as the elliptic dilogarithm thanks to Bloch's formula; see [17, 19] for the details. Note that

$$E_{3,\chi_{-3}}\left(-\frac{1}{3\tau}\right) = \frac{i\tau^3}{3\sqrt{3}}\,\tilde{E}_{3,\chi_{-3}}(\tau), \quad \tilde{E}_{3,\chi_{-3}}(\tau) = \frac{\eta(\tau)^9}{\eta(3\tau)^3} = 1 - 9\sum_{m,n=1}^{\infty}\left(\frac{-3}{n}\right)n^2q^{mn};$$

and, in addition, we have

$$\begin{aligned} \frac{1}{2\pi i}\,\frac{\mathrm{d}x/\mathrm{d}\tau}{x} = \frac{1}{2}\left(\frac{\eta(\tau)^2\eta(3\tau)^2}{\eta(2\tau)\eta(6\tau)}\right)^2 &= \frac{1}{18}\big(E_{1,\chi_{-3}}(\tau) - 4E_{1,\chi_{-3}}(4\tau)\big)^2 \\ &= \frac{1}{54\tau^2}\left(E_{1,\chi_{-3}}\left(-\frac{1}{12\tau}\right) - E_{1,\chi_{-3}}\left(-\frac{1}{3\tau}\right)\right)^2, \end{aligned}$$

where

$$E_{1,\chi_{-3}}(\tau) = 1 + 6\sum_{m,n=1}^{\infty}\left(\frac{-3}{m}\right)q^{mn}.$$

6. Modular computation for $W_5'(0)$ and $W_6'(0)$

As (partly) shown in [11] the density $p_4(x)$ can be parameterised as follows (we make a shift of τ by half):

$$p_4(x(\tau)) = -\operatorname{Re}\left(\frac{2i(1+6\tau+12\tau^2)}{\pi}\,p(\tau)\right),$$

where

$$p(\tau) = \frac{\eta(2\tau)^4\eta(6\tau)^4}{\eta(\tau)\eta(3\tau)\eta(4\tau)\eta(12\tau)} \quad \text{and} \quad x(\tau) = \left(\frac{2\eta(\tau)\eta(3\tau)\eta(4\tau)\eta(12\tau)}{\eta(2\tau)^2\eta(6\tau)^2}\right)^3.$$

The path for τ along the imaginary axis from 0 to $i/(2\sqrt{3})$ (or from $i\infty$ to $i/(2\sqrt{3})$) corresponds to x ranging from 0 to 2, while the path from $i/(2\sqrt{3})$ to $-1/4+i/(4\sqrt{3})$ along the arc centred at 0 corresponds to the real range $(2,4)$ for x. (The arc admits the parametrisation $\tau = e^{\pi i\theta}/(2\sqrt{3})$, $1/2 < \theta < 5/6$.) Note that $x(i/(2\sqrt{15})) = 1$ and

$$p_4(x(\tau)) = \begin{cases} -\dfrac{2i\cdot 6\tau}{\pi}\,p(\tau), & \text{for } \tau \text{ on the imaginary axis,} \\ -\dfrac{2i(1+6\tau+12\tau^2)}{\pi}\,p(\tau), & \text{for } \tau \text{ on the arc,} \end{cases}$$

and

$$-\frac{2i(1+6\tau+12\tau^2)}{\pi}\,p(\tau) = \frac{2\sqrt{16-x^2}}{\pi^2 x}\cdot {}_3F_2\left(\begin{matrix}\frac12,\frac12,\frac12\\ \frac56,\frac76\end{matrix}\;\middle|\;\frac{(16-x^2)^3}{108x^4}\right)$$

(this is a general form of [11, Theorem 4.9]). Formulas (1), (3) and (4) reduce the conjectural evaluations of $W_5'(0)$ and $W_6'(0)$ to the following ones:

$$\frac{7\zeta(3)}{2\pi^2} + L'(f_3;-1) \stackrel{?}{=} \frac{12}{\pi}\int_0^{1/(2\sqrt{15})} yp(iy)\log x(iy)\,\mathrm{d}x(iy)$$

and

$$\begin{aligned}\frac{7\zeta(3)}{2\pi^2} + 8L'(f_4;-1) &\stackrel{?}{=} \frac{12}{\pi}\int_0^{1/(2\sqrt{3})} yp(iy)\log x(iy)\,\mathrm{d}x(iy)\\ &\quad - \frac{12}{\pi^2}\int_0^{1/(2\sqrt{3})} yp(iy)x(iy)\cdot {}_3F_2\left(\begin{matrix}\frac12,\frac12,\frac12\\ \frac32,\frac32\end{matrix}\;\middle|\;\frac{x(iy)^2}{4}\right)\mathrm{d}x(iy).\end{aligned}$$

Furthermore, note that the Atkin–Lehner involutions $w_{12}\colon \tau\mapsto -1/(12\tau)$ and $w_6\colon \tau\mapsto(6\tau-5)/(12\tau-6)$ act on the modular function $x(\tau)$ as follows: $x(w_{12}\tau) = x(\tau)$ and $x(w_6\tau) = -8/x(\tau)$, and we also have $p(w_{12}\tau) = -\tau^2 p(\tau)$. The point $i/(2\sqrt{3})$ is fixed by w_{12}. Thus, the change of variable $y\mapsto 1/(12y)$ leads to

$$\int_0^{1/(2\sqrt{3})} yp(iy)\log x(iy)\,\mathrm{d}x(iy) = -\int_{1/(2\sqrt{3})}^{\infty} yp(iy)\log x(iy)\,\mathrm{d}x(iy).$$

7. Mahler measures related to a variation of random walk

In [23] the Mahler measures $\mathrm{m}(1+x_1+x_2)$ and $\mathrm{m}(1+x_1+x_2+x_3)$ are computed using the modular parametrisations of

$$\sum_{n=0}^{\infty} W_3(2n)z^n = \sum_{n=0}^{\infty}\mathrm{CT}\big((1+x_1+x_2)(1+x_1^{-1}+x_2^{-1})\big)^n z^n$$

and

$$\sum_{n=0}^{\infty} W_4(2n)z^n = \sum_{n=0}^{\infty}\mathrm{CT}\big((1+x_1+x_2+x_3)(1+x_1^{-1}+x_2^{-1}+x_3^{-1})\big)^n z^n,$$

where $\mathrm{CT}(L)$ denotes the constant term of a Laurent polynomial $L\in\mathbb{Z}[x_1^{\pm},x_2^{\pm},\dots]$. Note that the Picard–Fuchs linear differential equations for the two generating functions give rise to the ones for the densities $p_3(x)$ and $p_4(x)$ together with their explicit hypergeometric and modular expressions (see [11, eq. (3.2) and Remark 4.10]), though it remains unclear whether the latter information can be used to compute $W_N'(0)$ in (1) for $N=3,4$. This is itself an interesting question to not only assist in computing of $W_N'(0)$ for $N>4$ but also in relation with another famous conjecture of Boyd:

$$\mathrm{m}(1+x_1+x_2+x_3+x_2x_3) \overset{?}{=} -2L'(f_2;-1) = \frac{15^2}{4\pi^4}L(f_2;3) = 0.4839979734\ldots, \quad (6)$$

where $f_2(\tau)=\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)$.

In analogy with the case of linear Mahler measures, we define

$$\begin{aligned}\widetilde{W}(s) &= \iiint_{[0,1]^3} |1+e^{2\pi i\theta_1}+e^{2\pi i\theta_2}+e^{2\pi i\theta_3}+e^{2\pi i(\theta_2+\theta_3)}|^s \,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3 \\ &= Z(1+x_1+x_2+x_3+x_2x_3;s)\end{aligned}$$

as the s-th moment of a random 5-step walk for which the direction of the final step is completely determined by the two previous steps. Then the even moments

$$\begin{aligned}\widetilde{W}(2n) &= \mathrm{CT}\big((1+x_1+x_2+x_3+x_2x_3)(1+x_1^{-1}+x_2^{-1}+x_3^{-1}+(x_2x_3)^{-1})\big)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}^2\end{aligned}$$

satisfy a rather lengthy recurrence equation, which is equivalent to a Picard–Fuchs differential equation of order 4. The latter splits into the tensor product of two differential equations of order 2 and, with some effort, we obtain the following result.

Theorem 1. *We have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\widetilde{W}(2n)\left(\frac{t}{(4+t)(1+4t)}\right)^n \\ &\qquad = \frac{(4+t)(1+4t)}{4(1+4t+t^2)}\,{}_2F_1\left(\left.\begin{matrix}\frac12,\frac12\\1\end{matrix}\,\right|\,\frac{t(4+t)}{1+4t+t^2}\right)\cdot{}_2F_1\left(\left.\begin{matrix}\frac12,\frac12\\1\end{matrix}\,\right|\,\frac{t^2}{1+4t+t^2}\right)\end{aligned}$$

and, more generally,

$$\begin{aligned}&\frac{b}{(b+t)(1+bt)}\sum_{n=0}^{\infty}\left(\frac{t}{(b+t)(1+bt)}\right)^n\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}^2\left(\frac{b}{4}\right)^{2k} \\ &\qquad = {}_2F_1\left(\left.\begin{matrix}\frac12,\frac12\\1\end{matrix}\,\right|\,-t(b+t)\right)\cdot\frac{1}{(1+bt)^{1/2}}\,{}_2F_1\left(\left.\begin{matrix}\frac12,\frac12\\1\end{matrix}\,\right|\,-\frac{t^2}{1+bt}\right) \\ &\qquad = \frac{1}{1+bt+t^2}\,{}_2F_1\left(\left.\begin{matrix}\frac12,\frac12\\1\end{matrix}\,\right|\,\frac{t(b+t)}{1+bt+t^2}\right)\cdot{}_2F_1\left(\left.\begin{matrix}\frac12,\frac12\\1\end{matrix}\,\right|\,\frac{t^2}{1+bt+t^2}\right).\end{aligned}$$

Proof. Once a factorisation of this type is written down, it is a computational routine to prove it. In other words, a principal issue is discovering such a formula rather than proving it. Our original discovery of Theorem 1 involved a lot of experimental mathematics; however, we later realised that it is deducible from known formulae as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}^2 x^k &= \sum_{k=0}^{\infty} \binom{2k}{k}^2 x^k \sum_{m=0}^{\infty} \binom{k+m}{k}^2 z^{k+m} \\ &= \sum_{k=0}^{\infty} \binom{2k}{k}^2 (xz)^k \, {}_2F_1\left(\begin{matrix} k+1,\, k+1 \\ 1 \end{matrix} \,\middle|\, z\right) \\ &= \sum_{k=0}^{\infty} \binom{2k}{k}^2 \frac{(xz)^k}{(1-z)^{k+1}} \, {}_2F_1\left(\begin{matrix} -k,\, k+1 \\ 1 \end{matrix} \,\middle|\, -\frac{z}{1-z}\right) \\ &= \frac{1}{1-z} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{xz}{1-z}\right)^k \cdot P_k\left(\frac{1+z}{1-z}\right),\end{aligned}$$

where P_k denotes the k-th Legendre polynomial, and the latter generating function is a particular instance of the Bailey–Brafman formula [15, 34]. □

We remark that, using the general Bailey–Brafman formula and its generalisation from [29], the proof above extends to the factorisation of the two-variable generating functions

$$\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k}^2 \frac{(s)_k (1-s)_k}{k!^2} \, x^k$$

as well as of

$$\sum_{n=0}^{\infty} z^n \sum_{k} \binom{n}{2k}^2 \binom{2k}{k}^2 x^k \quad \text{and} \quad \sum_{n=0}^{\infty} z^n \sum_{k} \binom{n}{3k}^2 \frac{(3k)!}{k!^3} \, x^k,$$

and even of

$$\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k}^2 u_k \, x^k$$

for an Apéry-like sequence $u_0, u_1, u_2, \dots$.

Furthermore, we expect that Theorem 1 can lead to a hypergeometric expression for the density function $\widetilde{p}(x)$ (piecewise analytic, with finite support on the interval $0 < x < 5$), which is the inverse Mellin transform of $\widetilde{W}(s-1)$, hence to the Mahler measure evaluation

$$\mathrm{m}(1 + x_1 + x_2 + x_3 + x_2 x_3) = \widetilde{W}'(0) = \int_0^{\infty} \widetilde{p}(x) \log x \, \mathrm{d}x = \int_0^{5} \widetilde{p}(x) \log x \, \mathrm{d}x.$$

On the other hand, the reduction technique of Sections 3 and 4 suggests a different approach to computing $\widetilde{W}'(0)$, resulting in the following hypergeometric evaluation of the Mahler measure.

Theorem 2. *We have*

$$\mathrm{m}(1 + x_1 + x_2 + x_3 + x_2 x_3) = -\frac{1}{2\pi} \int_0^1 {}_2F_1\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2} \\ 1 \end{matrix} \,\middle|\, 1 - \frac{x^2}{16}\right) \log x \, \mathrm{d}x.$$

Proof. Define a related density $\widehat{p}(x)$ by

$$\int_0^4 x^s \widehat{p}(x)\,\mathrm{d}x = \widehat{W}(s) = \iint_{[0,1]^2} |1+e^{2\pi i\theta_2}+e^{2\pi i\theta_3}+e^{2\pi i(\theta_2+\theta_3)}|^s\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3$$
$$= W_2(s)^2 = \frac{\Gamma(1+s)^2}{\Gamma(1+s/2)^4}.$$

By an application of the Mellin transform calculus, we find that, for $0<x<4$,

$$\widehat{p}(x) = \frac{1}{2\pi}\cdot {}_2F_1\left(\begin{matrix}\frac12,\frac12\\1\end{matrix}\,\middle|\,1-\frac{x^2}{16}\right).$$

Then it follows from Lemma 1 that

$$\widetilde{W}'(0) = \int_1^4 \widehat{p}(x)\log x\,\mathrm{d}x = -\int_0^1 \widehat{p}(x)\log x\,\mathrm{d}x,$$

where we use the evaluation

$$\int_0^4 \widehat{p}(x)\log x\,\mathrm{d}x = \mathrm{m}(1+x_2+x_3+x_2x_3) = \mathrm{m}(1+x_2)+\mathrm{m}(1+x_3) = 0. \qquad \square$$

The above proof extends to the general formula

$$\mathrm{m}(1+bx_1+x_2+x_3+x_2x_3) = \log b\int_0^b \widehat{p}(x)\,\mathrm{d}x + \int_b^4 \widehat{p}(x)\log x\,\mathrm{d}x$$
$$= \frac{1}{2\pi}\int_0^b {}_2F_1\left(\begin{matrix}\frac12,\frac12\\1\end{matrix}\,\middle|\,1-\frac{x^2}{16}\right)\log\frac{b}{x}\,\mathrm{d}x$$

for $0<b\le 4$. A related computation

$$\mathrm{m}(1+bx_1+x_2+x_3+x_2x_3) = \log b + \frac{8}{\pi^2}\int_b^4 \frac{\arccos(b/x)\,\log(x/(2\sqrt{b}))}{\sqrt{16-x^2}}\,\mathrm{d}x$$

valid for $0<b\le 4$ was given by J. Wan [27]; he also pointed out that $\mathrm{m}(1+bx_1+x_2+x_3+x_2x_3)=\log b$ for $b>4$ follows from Jensen's formula.

The left-hand side of another Mahler measure conjecture [13]

$$\mathrm{m}((1+x_1)^2+x_2+x_3) \overset{?}{=} -L'(\tilde{f}_2;-1) = \frac{72}{\pi^4}\,L(\tilde{f}_2;3) = 0.7025655062\ldots,$$

where $\tilde{f}_2(\tau)=\eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)$ is a cusp form of level 24, can be treated by a similar reduction, using that the densities for $(1+x_1)^2$ and x_2+x_3 are $p_2(t^{1/2})/(2t^{1/2})$ on $[0,4]$ and $p_2(t)$ on $[0,2]$, respectively. The final result is the elegant formula

$$\mathrm{m}((1+x_1)^2+x_2+x_3) = \frac{2G}{\pi} + \frac{2}{\pi^2}\int_0^1 \arcsin(1-x)\,\arcsin x\,\frac{\mathrm{d}x}{x}, \tag{7}$$

where G is Catalan's constant, and, with some further work, we can express the right-hand side hypergeometrically.

Theorem 3. *We have*

$$\mathrm{m}((1+x_1)^2+x_2+x_3) = \frac{8\Gamma(\frac34)^2}{\pi^{5/2}}\,{}_5F_4\left(\begin{matrix}\frac14,\frac14,\frac14,\frac34,\frac34\\ \frac12,\frac54,\frac54,\frac54\end{matrix}\,\middle|\,\frac14\right) + \frac{\Gamma(\frac14)^2}{54\pi^{5/2}}\,{}_5F_4\left(\begin{matrix}\frac34,\frac34,\frac34,\frac54,\frac54\\ \frac32,\frac74,\frac74,\frac74\end{matrix}\,\middle|\,\frac14\right).$$

Proof. Notice that, for $0<x<1$,

$$\arcsin(1-x) = \frac{\pi}{2} - \arccos(1-x) = \frac{\pi}{2} - \sqrt{2x}\,{}_2F_1\left(\begin{matrix}\frac12,\frac12\\ \frac32\end{matrix}\,\middle|\,\frac{x}{2}\right),$$

and that, for $n>-1/2$,

$$\int_0^1 x^{n-1/2}\arcsin x\,\mathrm{d}x = \frac{\sqrt{\pi}}{2n+1}\left(\sqrt{\pi} - \frac{\Gamma(\frac{n}{2}+\frac34)}{\Gamma(\frac{n}{2}+\frac54)}\right).$$

Therefore,

$$\int_0^1 \arcsin(1-x)\,\arcsin x\,\frac{\mathrm{d}x}{x} = \frac{\pi}{2}\int_0^1 \arcsin x\,\frac{\mathrm{d}x}{x} - \pi\sqrt{2}\sum_{n=0}^{\infty}\frac{(\frac12)_n^2}{n!\,(\frac32)_n(2n+1)}\,\frac{1}{2^n} + \sqrt{2\pi}\sum_{n=0}^{\infty}\frac{(\frac12)_n^2\Gamma(\frac{n}{2}+\frac34)}{n!\,(\frac32)_n(2n+1)\,\Gamma(\frac{n}{2}+\frac54)}\,\frac{1}{2^n}.$$

From this and (7) we deduce

$$\mathrm{m}((1+x_1)^2+x_2+x_3) = \frac{2G}{\pi} + \frac{\log 2}{2} - \frac{2\sqrt2}{\pi}\,{}_3F_2\left(\begin{matrix}\frac12,\frac12,\frac12\\ \frac32,\frac32\end{matrix}\,\middle|\,\frac12\right) + \frac{8\sqrt2\,\Gamma(\frac34)}{\pi^{3/2}\Gamma(\frac14)}\,{}_5F_4\left(\begin{matrix}\frac14,\frac14,\frac14,\frac34,\frac34\\ \frac12,\frac54,\frac54,\frac54\end{matrix}\,\middle|\,\frac14\right) + \frac{\sqrt2\,\Gamma(\frac14)}{54\pi^{3/2}\Gamma(\frac34)}\,{}_5F_4\left(\begin{matrix}\frac34,\frac34,\frac34,\frac54,\frac54\\ \frac32,\frac74,\frac74,\frac74\end{matrix}\,\middle|\,\frac14\right).$$

It remains to use

$$G + \frac14\pi\log 2 = \sqrt2\,{}_3F_2\left(\begin{matrix}\frac12,\frac12,\frac12\\ \frac32,\frac32\end{matrix}\,\middle|\,\frac12\right)$$

(see [1, Entry 30]) and $\Gamma(\frac14)\Gamma(\frac34) = \pi\sqrt2$. □

8. Conclusion

A goal of this final section is to highlight relevance for and links with other research and open problems.

The (hypergeometric) factorisation in Theorem 1 and similar results outlined after its proof are part of a general phenomenon of arithmetic differential equations of order 4. These are the first instances "beyond modularity" in the sense that arithmetic differential equations of order 2 and 3 are always supplied by modular parametrisation. In order 4, we have to distinguish two particular novel situations (though our knowledge about either is imperfect and incomplete): (the Zariski closure of) the monodromy group is the orthogonal group $O_4 \simeq O_{2,2}$ of dimension 6 or the symplectic group Sp_4 of dimension 10. The example given in Theorem 1 corresponds to the first (orthogonal) situation: on the level of Lie groups, $O_{2,2}$ can be realised as the tensor product of two copies of SL_2 (or GL_2). There is a limited amount of further examples of this type [21, 29, 33] though we expect that all underlying Picard–Fuchs differential equations with such monodromy can be represented as tensor products of two arithmetic differential equations of order 2. There is a natural hypergeometric production of such orthogonal cases using Orr-type formulae (see [18, 28]) but there are plenty of other cases coming from classical work of W. N. Bailey and its recent generalisations [29, 34]. Many such cases, mostly forecast by Sun [25], are still awaiting their explicit factorisation. Though these situations do not cover symplectic monodromy instances, they can still be viewed as an intermediate step between classical modularity and Sp_4: the antisymmetric square of the latter happens to be $O_5 \simeq O_{3,2}$ (see [4]).

More in the direction of three-variable Mahler measure, the conjectural evaluation in (6) and Theorem 2 brings us to the expectation

$$\frac{1}{2\pi}\int_0^1 {}_2F_1\left(\begin{matrix}\frac12,\frac12\\1\end{matrix}\,\middle|\,1-\frac{x^2}{16}\right)\log x\,\mathrm{d}x \overset{?}{=} 2L'(f_2;-1). \tag{8}$$

This one highly resembles the evaluation

$$\frac12\int_0^1 {}_2F_1\left(\begin{matrix}\frac12,\frac12\\1\end{matrix}\,\middle|\,\frac{x^2}{16}\right)\mathrm{d}x = \frac12\cdot{}_3F_2\left(\begin{matrix}\frac12,\frac12,\frac12\\1,\frac32\end{matrix}\,\middle|\,\frac{1}{16}\right) = 2L'(f_2;0) \tag{9}$$

established in [22]. The related modular parametrisation

$$x = x(\tau) = 16\left(\frac{\eta(\tau)\eta(4\tau)^2}{\eta(2\tau)^3}\right)^4$$

corresponds to

$$1-\frac{x^2}{16} = \left(\frac{\eta(\tau)^2\eta(4\tau)}{\eta(2\tau)^3}\right)^8,$$
$$F\left(\frac{x^2}{16}\right) = \frac{\eta(2\tau)^{10}}{\eta(\tau)^4\eta(4\tau)^4} \quad\text{and}\quad F\left(1-\frac{x^2}{16}\right) = -2i\tau F\left(\frac{x^2}{16}\right),$$

where F denotes the corresponding ${}_2F_1$ hypergeometric series. Note that x ranges from 0 to 4 when τ runs from $i\infty$ to 0 along the imaginary axis; however, the point $\tau_0 = i\,0.8774376613482\ldots$, at which $x(\tau_0)=1$, is not a quadratic irrationality. Furthermore, H. Cohen [16] observes another step in the ladder (9), (8):

$$\begin{aligned}\frac{6}{\pi^2}\int_0^1 {}_2F_1\left(\begin{matrix}\frac12,\frac12\\1\end{matrix}\,\middle|\,\frac{x^2}{16}\right)\log^2 x\,\mathrm{d}x \overset{?}{=} 2L'(f_2;-2) &= \frac{3\cdot 15^3}{8\pi^6}\,L(f_2;4)\\ &= 1.2165632526\ldots,\end{aligned} \tag{10}$$

though not linked to a particular Mahler measure.

The expression in Theorem 3 is somewhat different from the one in Theorem 2, and resembles the hypergeometric evaluation of the L-value

$$\begin{aligned}-L'(\hat{f}_2;-1)&=\frac{128}{\pi^4}L(\hat{f}_2;3)\\&=\frac{\Gamma(\frac14)^2}{6\sqrt{2}\pi^{5/2}}\,{}_4F_3\left(\begin{matrix}1,\,1,\,1,\,\frac12\\ \frac74,\,\frac32,\,\frac32\end{matrix}\,\middle|\,1\right)+\frac{4\Gamma(\frac34)^2}{\sqrt{2}\pi^{5/2}}\,{}_4F_3\left(\begin{matrix}1,\,1,\,1,\,\frac12\\ \frac54,\,\frac32,\,\frac32\end{matrix}\,\middle|\,1\right)\\&\quad+\frac{\Gamma(\frac14)^2}{2\sqrt{2}\pi^{5/2}}\,{}_4F_3\left(\begin{matrix}1,\,1,\,1,\,\frac12\\ \frac34,\,\frac32,\,\frac32\end{matrix}\,\middle|\,1\right),\end{aligned}$$

where $\hat{f}_2(\tau)=\eta(4\tau)^2\eta(8\tau)^2$ is a cusp form of level 32, obtained in [32, Theorem 3].

Finally, we remark that the integral

$$W_3'(0)=\int_0^3\log x\,\mathrm{d}P_3(x)=\log 3-\int_0^3 P_3(x)\,\frac{\mathrm{d}x}{x}$$

in the notation of Section 5, with $P_3(x)$ related to eta quotients, is visually linked to the following result in [7] (also discussed in greater generality in [2, 26])

$$\int_0^1\frac19\left(1-\frac{\eta(\tau)^9}{\eta(3\tau)^3}\right)\frac{\mathrm{d}q}{q}=\lim_{q\to1^-}\sum_{m,n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{n}{m}\,q^{mn}=L'(\chi_{-3};-1).$$

However, apart from the fact that the two quantities coincide we could not find a direct link between the two integrals.

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