Eulerian Numbers

A First Glance

23 April 2007

Contents

1 Abstract Definition

Let $\sigma \in S_n$ be a permutation. An ascent of σ is an occurrence of $\sigma(j) < \sigma(j+1)$ for $j = 1, \ldots, n-1$. The Eulerian number $\left\langle {n \atop k} \right\rangle$ is the number of permutations in S_n with exactly k ascents.

Denote the permutation $\sigma \in S_n$ by $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$. For example $[4, 2, 1, 3]$ is the permutation σ in S_4 with $\sigma(1) = 4$, $\sigma(2) = 2$, $\sigma(3) = 1$, $\sigma(4) = 3$. This has only one ascent $[1, 3]$. Likewise

$$\begin{array}{ll} [5,1,3,4,2] & \text{has 2 ascents,} \\ [2,3,4,1,5] & \text{has 3 ascents.} \end{array}$$

Clearly,

$$\sum_k \left\langle {n \atop k} \right\rangle = n!,$$

since summing over k makes us count all the permutations in S_n regardless of their ascent.

They also observe the following symmetry

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle,$$

which captures the fact that Eulerian numbers could as well have been defined in terms of descents. This is seen by simply reversing the bracketed representation $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$ turning ascents into descents and vice versa.

The Eulerian numbers appear as sequence `A008292` in [Sloane, 2007]. The notation as well as the choice of indices varies across the literature. A particularly often used alternative is

$$A_{n,k} \triangleq \left\langle {n \atop k+1} \right\rangle.$$

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2 Recurrence

The Eulerian numbers satisfy the following defining recurrence relation

$$\left\langle {n \atop k} \right\rangle = (k+1) \left\langle {n-1 \atop k} \right\rangle + (n-k) \left\langle {n-1 \atop k-1} \right\rangle, \quad \left\langle {0 \atop k} \right\rangle = \delta_{k=0},$$

with the understanding that $\left\langle {n \atop k} \right\rangle = 0$ for $n < 0$.

Proof. First, note that every permutation in S_n can be constructed from a permutation $\sigma \in S_{n-1}$ by "inserting" n into the bracketed list $[\sigma(1), ..., \sigma(n-1)]$. So either n is inserted at an end, or we have

$$[..., a, n, b, ...].$$

Since $a < n > b$ this account for one ascent. In any way, the insertion either leaves the number of ascents invariant or increases it by one.
To get an element with k ascents in S_n while leaving the number of ascents invariant, we have to start with one of the $\left\langle {n-1 \atop k} \right\rangle$ permutations in S_{n-1} with exactly k ascents. We have to insert n either at one of the k places where an ascent occured or at the beginning. This amounts to

$$(k+1) \left\langle {n-1 \atop k} \right\rangle$$

possibilities.
When increasing the number of ascents we have to start with one of the $\left\langle {n-1 \atop k-1} \right\rangle$ permutations with $k-1$ ascents, and insert n either at one of the $(n-1)-1-(k-1)$ places of an descent or at the very end. That makes

$$(n-k) \left\langle {n-1 \atop k-1} \right\rangle$$

possibilities. □

We can arrange the Eulerian numbers in a triangle like we are used to for the binomial coefficients. Compare Pascal's triangle and Euler's triangle.

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array} \qquad \begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&4&&1&& \\ &1&&11&&11&&1& \\ 1&&26&&66&&26&&1 \end{array}$$

Note that the n-th row indeed sums to $n!$.

3 Counting Points in Hypercubes

How many integer points are in the hypercube $[1, x]^n$? Fair enough, you might say, that's trivial for it's just x^n many. Let's, however, count these points in a different combinatorial fashion, and get an identity for x^s in this way. We have already seen that the basis $\{1, x, x^2, ...\}$ for polynomials often needs to be replaced by more convenient basis especially in the context of discrete calculus, where we encountered the falling factorials $\{1, x, x^{\underline{2}}, ...\}$.

$n = 1$.

We have to choose i such that $1 \leqslant i \leqslant x$. Clearly, there are $\binom{x}{1}$ many such possibilities, and we get the trivial

$$x^1 = \binom{x}{1}.$$

$n = 2$.

Now, the points in the hypercube are given by pairs (i, j) such that $1 \leqslant i, j \leqslant x$. Then there are two cases.

$$i \leqslant j, \quad j < i.$$

The first case makes up for $\binom{x+1}{2}$ possibilities while the second contributes $\binom{x}{2}$ many. To see the former just note that the number of (i, j) such that $1 \leqslant i \leqslant j \leqslant x$ is the same as the number of those (i, j) for which $1 \leqslant i < j \leqslant x + 1$. We thus have

$$x^2 = \binom{x}{2} + \binom{x+1}{2}.$$

$n = 3.$

Here we have to count triples (i, j, k) for which $1 \leqslant i, j, k \leqslant x$. Imitating what we did in the previous case we end up with the following 6 possibilities.

$$\begin{array}{ccccc} i & \leqslant & j & \leqslant & k \\ i & \leqslant & k & < & j \\ j & < & i & \leqslant & k \\ j & \leqslant & k & < & i \\ k & < & i & \leqslant & j \\ k & < & j & < & i \end{array}$$

The first corresponds to $\binom{x+2}{3}$ possibilities while the next four each provide $\binom{x+1}{3}$ possibilities. The last one contributes another $\binom{x}{3}$. Hence

$$x^3 = \binom{x}{3} + 4\binom{x+1}{3} + \binom{x+2}{3}.$$

Note that the 6 cases correspond to the permutations of $\{i, j, k\}$.

General Case.

The points are $1 \leqslant j_1, \ldots, j_n \leqslant x$. Again, by considering the permutations σ of $\{1, \ldots, n\}$ we are lead to distinct $n!$ cases. But how to decide whether to use $\leqslant$ or $<$? One choice is to take $j_{\sigma(k)} \leqslant j_{\sigma(k+1)}$ if $\sigma(k) < \sigma(k+1)$ and $j_{\sigma(k)} < j_{\sigma(k+1)}$ otherwise. That's what we did in the case $n = 3$. If you think about it for a second you'll see that by doing so we ensure that we cover all possibilities and that our cases don't overlap.
Now, pick some permutation σ. Say there are λ occurences of $\leqslant$. Then σ contributes $\binom{x+\lambda}{n}$ possibilities. So how often does a $\leqslant$ appear? A $\leqslant$ occurrs whenever $\sigma(k) < \sigma(k+1)$, that is for each ascent of σ.
This immediately leads to

$$x^n = \sum_{\lambda=0}^{n-1} \left\langle {n \atop \lambda} \right\rangle \binom{x+\lambda}{n}.$$

Note that

$$\binom{x}{n} = \frac{x^{\overline{n}}}{n!}$$

gives a close relation to the rising factorials, and that

$$\binom{x}{n}, \binom{x+1}{n}, \ldots, \binom{x+n-1}{n}$$

form a basis of the polynomials of degree less or equal then n. This follows easily from our representation of x^n, and from

$$\Delta_x \binom{x}{n} = \binom{x+1}{n} - \binom{x}{n} = \frac{\Delta_x x^{\overline{n}}}{n!} = \frac{x^{\overline{n-1}}}{(n-1)!} = \binom{x}{n-1},$$

which also is immediate from the recurrence of the binomial coefficients (generalized to non-integral x).

Example 1. Let's use our result to find a formula for

$$\sum_{x=1}^{N} x^n.$$

Clearly, from above

$$\Delta_x \binom{x+\lambda}{n} = \binom{x+\lambda}{n-1},$$

and hence

$$\sum_{x=1}^{N} \binom{x+\lambda}{n} = \binom{N+1+\lambda}{n+1} - \binom{1+\lambda}{n+1}.$$

Finally,

$$\sum_{x=1}^{N} x^n = \sum_{\lambda=0}^{n-1} \left\langle {n \atop \lambda} \right\rangle \sum_{x=1}^{N} \binom{x+\lambda}{n} = \sum_{\lambda=0}^{n-1} \left\langle {n \atop \lambda} \right\rangle \binom{N+\lambda+1}{n+1}.$$

4 Differentiating the Geometric Series

We have encountered the operator $x\ D$, which sometimes is called the Euler operator, when studying generating functions. If

$$\{a_n\} \overset{\text{ogf}}{\longleftrightarrow} F, \quad \text{then} \quad \{n\, a_n\} \overset{\text{ogf}}{\longleftrightarrow} (x\, D)\, F.$$

Let's apply $x\, D$ to the most basic generating function, namely the geometric series.

$$\begin{aligned}
(x\, D) \frac{1}{1-x} &= \frac{x}{(1-x)^2} \\
(x\, D)^2 \frac{1}{1-x} &= \frac{x}{(1-x)^3} (1+x) \\
(x\, D)^3 \frac{1}{1-x} &= \frac{x}{(1-x)^4} \left(1+4x+x^2\right) \\
(x\, D)^4 \frac{1}{1-x} &= \frac{x}{(1-x)^5} \left(1+11x+11x^2+x^3\right) \\
&\vdots \\
(x\, D)^n \frac{1}{1-x} &= \frac{x}{(1-x)^{n+1}} \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k.
\end{aligned}$$

Proof. It follows from induction that

$$(x\, D)^n = \sum_k \left\{ {n \atop k} \right\} x^k\, D^k.$$

Clearly,

$$D^n \frac{1}{1-x} = (-1)^n \frac{n!}{(1-x)^{n+1}}.$$

Now proceed as in [Stopple, 2003]. □

5 Occurrence in Probability Theory

Recall that $\sum_k \left\langle {n \atop k} \right\rangle = n!$ so normalizing the Eulerian numbers gives probability weights. They appear in the following way. Let X_j be independent random variables with uniform distribution on [0, 1]. Then the Eulerian numbers give the probability that sums of these random variables take values in intervals of unit length.

$$\frac{1}{n!} \left\langle {n \atop k} \right\rangle = \mathbb{P}\left(\sum_{j=1}^{n} X_j \in [k, k+1] \right).$$

That may be checked by showing that these probabilities obey the same recurrence. By the central limit theorem this identity implies that the probability weigths induced by the Eulerian numbers approach a normal distribution (after proper normalizing). To be precise, by the central limit theorem

$$\lim_{n\to\infty} \mathbb{P}\left(\sum_{j=1}^{n} \frac{X_j - 1/2}{\sqrt{n/12}} \leqslant x \right) = \Phi(x).$$

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This justifies the following estimate

$$\left\langle {n \atop k} \right\rangle \approx n! \left(\Phi\left(\frac{k+1-n/2}{\sqrt{n/12}} \right) - \Phi\left(\frac{k-n/2}{\sqrt{n/12}} \right) \right),$$

which is not to be read in an asymptotic sense.

6 Relations

We have, see [Stopple, 2003], the following explicit formula due to Euler himself,

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n.$$

This identity allows us to find closed expressions for the Eulerian numbers when the number of ascents k is fixed. For example

$$\begin{aligned} \left\langle {n \atop 1} \right\rangle &= 2^n - n - 1 \\ \left\langle {n \atop 2} \right\rangle &= 3^n - (n+1)\,2^n + \binom{n+1}{2}. \end{aligned}$$

The identity also provides us with the asymptotics for fixed k, namely

$$\left\langle {n \atop k} \right\rangle \sim (k+1)^n, \quad \text{as } n \to \infty.$$

Let B_n be the Bernoulli numbers (that is $B_n/n!$ are the coefficients of the Taylor series of $x/(\exp(x)-1)$). Then for $n \geqslant 1$,

$$\sum_k (-1)^{k+1} \left\langle {n \atop k} \right\rangle = 2^{n+1} \left(2^{n+1}-1\right) \frac{B_{n+1}}{n+1}.$$

For details see [Stopple, 2003]. Note that this is also a connection to the Riemann ζ-function since

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}.$$

7 Generating Functions

For $n \geqslant 1$ we have

$$\left\langle {n \atop k} \right\rangle = \left[\frac{x^n\, y^{k+1}}{n!} \right] \left(\frac{1-y}{1-y\,e^{(1-y)x}} \right),$$

or more precisely, see [Carlitz, 1976],

$$1 + \sum_{n,k \geqslant 1} A_{n,k} \frac{x^n\, y^k}{n!} = \frac{1-y}{1-y\,e^{(1-y)x}}.$$

We also have the following generating function, cf. [Carlitz, 1978],

$$\left\langle {r+s+1 \atop r} \right\rangle = \left[\frac{x^r\, y^s}{(r+s+1)!} \right] \frac{e^x - e^y}{x\,e^y - y\,e^x}.$$

The numbers on the left-hand side are used to define the symmetrically indexed Eulerian numbers

$$A_{[r,s]} \triangleq \left\langle {r+s+1 \atop r} \right\rangle = \left\langle {r+s+1 \atop s} \right\rangle = A_{[s,r]}.$$

Then

$$\frac{e^x - e^y}{x\,e^y - y\,e^x} = \sum_{r,s \geqslant 0} A_{[r,s]} \frac{x^r\, y^s}{(r+s+1)!}.$$

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