Log-sine integrals: Demonstration

Basic usage

As a first step, the log-sine integral package needs to be loaded in some way.

Ultimately, this step is hoped to become unnecessary if the functionality is integrated into the core of SAGE.

```
attach "/home/armin/docs/math/sage/logsine.sage"
```

Log-sine integrals can now be entered as in the examples below.

If the option *typeset* is selected for this notebook then the resulting output matches the notation used in mathematical papers.

```
ls(5,1,pi/3)
```

$$\mathrm{Ls}_5^{(1)}(\tfrac{1}{3}\,\pi)$$

```
ls(5,0,pi/3)
```

$$\mathrm{Ls}_5(\tfrac{1}{3}\,\pi)$$

The main functionality of this package is the conversion of log-sine integrals to polylogarithms. This is achieved with the command *lstoli* as illustrated in the next few examples:

```
lstoli(-ls(4,2,pi))
```

$$\frac{3}{2}\,\pi\zeta(3)$$

```
lstoli(ls(4,1,pi))
```

$$-\frac{11}{720}\,\pi^4 - 2\,\mathrm{Gl}_{3,1}(\pi)$$

```
lstoli(-ls(5,1,pi))
```

$$-\frac{1}{4}\,\pi^2\zeta(3) + \frac{105}{32}\,\zeta(5) - 6\,\mathrm{Cl}_{3,1,1}(\pi)$$

```
lstoli(-ls(6,1,pi))
```

$$-\frac{3}{1120}\,\pi^6 + 3\,\zeta(3)^2 + 24\,\mathrm{Gl}_{3,1,1,1}(\pi) - 18\,\mathrm{Gl}_{5,1}(\pi)$$

```
lstoli(ls(5,1,pi/3))
```

$\frac{1}{12}\pi^2\zeta(3) + \frac{3}{2}\pi\mathrm{Cl}_4(\frac{1}{3}\pi) - \frac{19}{4}\zeta(5)$

```
lstoli(ls(5,0,pi/3))
```

$-\frac{1543}{19440}\pi^5 + 6\,\mathrm{Gl}_{4,1}(\frac{1}{3}\pi)$

```
for n in [2..8]:
    view(ls(n,0,pi) == lstoli(ls(n,0,pi)))
```

$\mathrm{Ls}_2(\pi) = 0$
$\mathrm{Ls}_3(\pi) = -\frac{1}{12}\pi^3$
$\mathrm{Ls}_4(\pi) = \frac{3}{2}\pi\zeta(3)$
$\mathrm{Ls}_5(\pi) = -\frac{19}{240}\pi^5$
$\mathrm{Ls}_6(\pi) = \frac{5}{4}\pi^3\zeta(3) + \frac{45}{2}\pi\zeta(5)$
$\mathrm{Ls}_7(\pi) = -\frac{275}{1344}\pi^7 - \frac{45}{2}\pi\zeta(3)^2$
$\mathrm{Ls}_8(\pi) = \frac{133}{32}\pi^5\zeta(3) + \frac{315}{8}\pi^3\zeta(5) + \frac{2835}{4}\pi\zeta(7)$

```
for n in [2..7]:
    view(ls(n,0,pi/3) == lstoli(ls(n,0,pi/3)))
```

$\mathrm{Ls}_2(\frac{1}{3}\pi) = \mathrm{Cl}_2(\frac{1}{3}\pi)$
$\mathrm{Ls}_3(\frac{1}{3}\pi) = -\frac{7}{108}\pi^3$
$\mathrm{Ls}_4(\frac{1}{3}\pi) = \frac{1}{2}\pi\zeta(3) + \frac{9}{2}\mathrm{Cl}_4(\frac{1}{3}\pi)$
$\mathrm{Ls}_5(\frac{1}{3}\pi) = -\frac{1543}{19440}\pi^5 + 6\,\mathrm{Gl}_{4,1}(\frac{1}{3}\pi)$
$\mathrm{Ls}_6(\frac{1}{3}\pi) = \frac{35}{36}\pi^3\zeta(3) + \frac{15}{2}\pi\zeta(5) + \frac{135}{2}\mathrm{Cl}_6(\frac{1}{3}\pi)$
$\mathrm{Ls}_7(\frac{1}{3}\pi) = -\frac{74369}{326592}\pi^7 - \frac{15}{2}\pi\zeta(3)^2 + 135\,\mathrm{Gl}_{6,1}(\frac{1}{3}\pi)$

```
lstoli(ls(6,3,pi/3)-2*ls(6,1,pi/3))
```

$\frac{313}{204120}\pi^6$

The argument of the log-sine integral may be symbolic:

```
assume(0<x, x<2*pi)
lstoli(ls(4,1,x))
```

$\frac{1}{180}\pi^4 - \frac{1}{8}\pi^2x^2 + \frac{1}{6}\pi x^3 - \frac{1}{16}x^4 - 2\,x\mathrm{Gl}_{2,1}(x) - 2\,\mathrm{Gl}_{3,1}(x)$

```
lstoli(ls(6,4,x))
```

$x^4\mathrm{Cl}_2(x) + 4\,x^3\mathrm{Cl}_3(x) - 12\,x^2\mathrm{Cl}_4(x) - 24\,x\mathrm{Cl}_5(x) + 24\,\mathrm{Cl}_6(x)$

```
assume(0<x, x<2*pi)
for n in [3..4]:
    for k in [0..n-2]:
        view(ls(n,k,x) == lstoli(ls(n,k,x)))
```

$\mathrm{Ls}_3(x) = -\frac{1}{4}\pi^2x + \frac{1}{4}\pi x^2 - \frac{1}{12}x^3 - 2\,\mathrm{Gl}_{2,1}(x)$

$$\mathrm{Ls}_3^{(1)}(x) = x\mathrm{Cl}_2(x) - \zeta(3) + \mathrm{Cl}_3(x)$$
$$\mathrm{Ls}_4(x) = \frac{3}{4}\,\pi^2\mathrm{Cl}_2(x) - \frac{3}{2}\,\pi x\mathrm{Cl}_2(x) + \frac{3}{4}\,x^2\mathrm{Cl}_2(x) + \frac{3}{2}\,\pi\zeta(3) - \frac{3}{2}\,\pi\mathrm{Cl}_3(x) + \frac{3}{2}\,x($$
$$\mathrm{Ls}_4^{(1)}(x) = \frac{1}{180}\,\pi^4 - \frac{1}{8}\,\pi^2x^2 + \frac{1}{6}\,\pi x^3 - \frac{1}{16}\,x^4 - 2\,x\mathrm{Gl}_{2,1}(x) - 2\,\mathrm{Gl}_{3,1}(x)$$
$$\mathrm{Ls}_4^{(2)}(x) = x^2\mathrm{Cl}_2(x) + 2\,x\mathrm{Cl}_3(x) - 2\,\mathrm{Cl}_4(x)$$